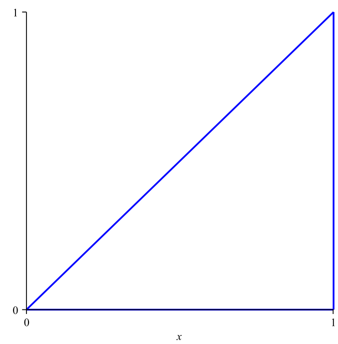
Calculus 3 - Cylindrical and Spherical Coordinates

In Calculus 2 we introduced polar coordinates

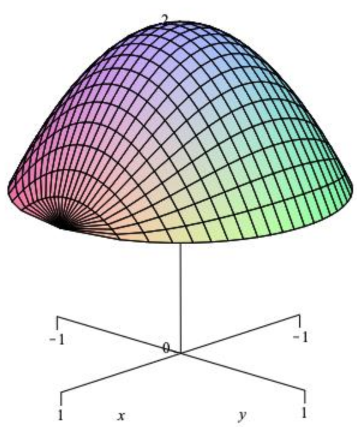
$$x = r\cos\theta, \quad y = r\sin\theta, \tag{1}$$

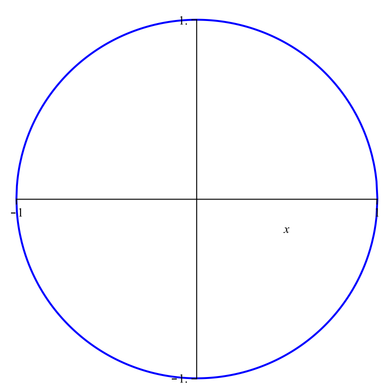
and

$$x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x}. \tag{2}$$



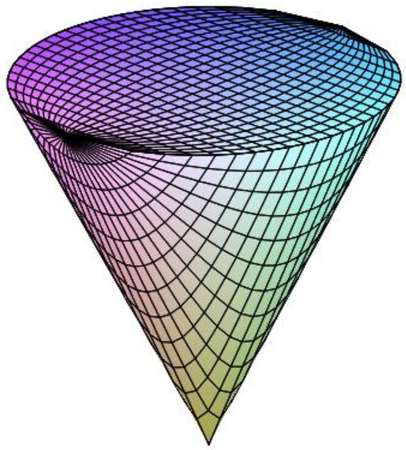
We saw that some double integrals would be easier in this coordinate system. For example, the volume under the paraboloid $z = 2 - x^2 - y^2$ and inside the cylinder $x^2 + y^2 = 1$, for $z \geq 0$

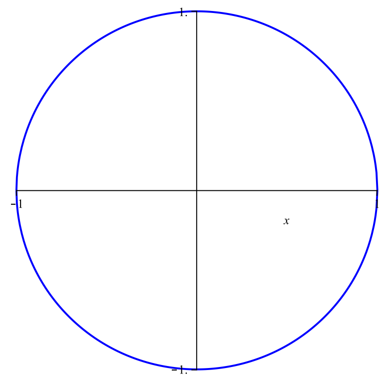




$$V = \int_0^{2\pi} \int_0^1 (2 - r^2) r \, dr \, d\theta \tag{3}$$

Earlier this week we introduced triple integrals. For example, set up the triple integral for the volume bound by the cones $z = \sqrt{x^2 + y^2}$ and the plane $z = 1$





In Cartesian coordinates, the setup would be

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} F(x, y, z) dz\, dy\, dx. \tag{4}$$

Maybe there are ”polar” coordinates in 3D would make triple integrals easier. Here we introduce two types of polar coordinates:

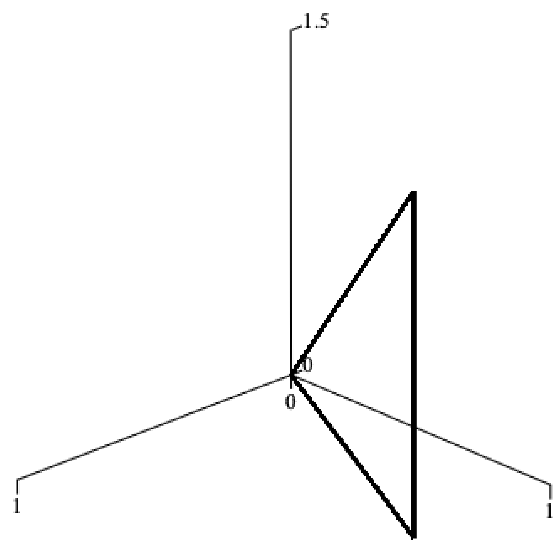
- cylindrical polar coordinates
- spherical polar coordinates

They both have their advantages and disadvantages which we will discover.

Cylindrical Polar Coordinates

These are similar to the usual polar coordinates that are in 2D but we simply add z to extend to 3D so

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{5}$$



Let us consider a number of quadratic surfaces.

Sphere

$$x^2 + y^2 + z^2 = a^2 \quad \Rightarrow \quad r^2 + z^2 = a^2 \tag{6}$$

Hyperboloid of 1 Sheet

$$x^2 + y^2 - z^2 = 1 \quad \Rightarrow \quad r^2 - z^2 = 1 \tag{7}$$

Hyperboloid of 2 Sheets

$$-x^2 - y^2 + z^2 = 1 \quad \Rightarrow \quad -r^2 + z^2 = 1 \tag{8}$$

Cone

$$x^2 + y^2 = z^2 \quad \Rightarrow \quad r^2 = z^2 \text{ or } z = \pm r \tag{9}$$

Paraboloid

$$z = x^2 + y^2 \quad \Rightarrow \quad z = r^2 \tag{10}$$

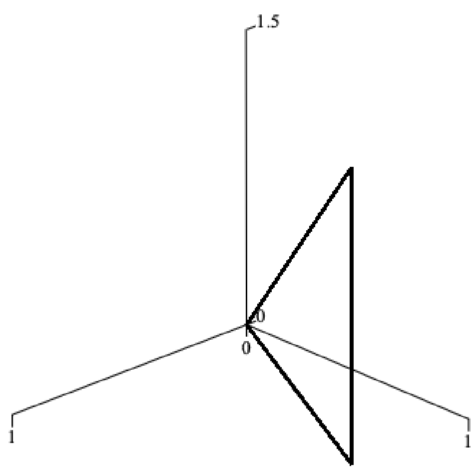
Notice that each of these equations are independent of θ so draw the picture in the rz plane and rotate $0 \to 2\pi$.

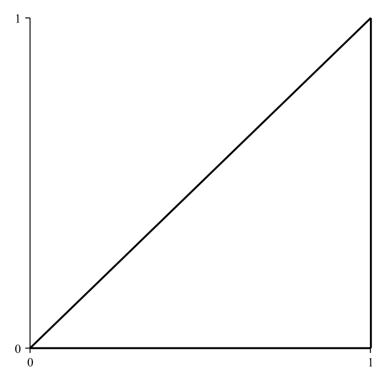
Plane

$$x + y + z = 1 \quad \Rightarrow \quad r\cos\theta + r\sin\theta + z = 1 \tag{11}$$

This one is better in Cartesian!

Spherical Polar Coordinates





We see that

$$\sin\phi = \frac{r}{\rho}, \quad \cos\phi = \frac{z}{\rho}, \tag{12}$$

or

$$r = \rho \sin \phi \quad z = \rho \cos \phi \tag{13}$$

and since

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z. \tag{14}$$

then

$$x = \rho \cos \theta \sin \phi, \quad y = \rho \sin \theta \sin \phi, \quad z = \rho \cos \phi. \tag{15}$$

Now, we see that

$$x^2 + y^2 = \rho^2 \sin^2 \phi \quad \text{so} \quad \sqrt{x^2 + y^2} = \rho \sin \phi \tag{16}$$

and from (15) and (16)

$$\tan \theta = \frac{y}{x}, \quad \tan \phi = \frac{\sqrt{x^2 + y^2}}{z}, \quad x^2 + y^2 + z^2 = \rho^2. \tag{17}$$

Some Surfaces

Sphere

$$x^2 + y^2 + z^2 = a^2 \quad \Rightarrow \quad \rho^2 = a^2 \ \ or \ \ \rho = a \tag{18}$$

Hyperboloid of 1 Sheet

$$x^2 + y^2 - z^2 = 1, \quad \Rightarrow \quad \rho^2 \sin^2 \phi - \rho^2 \cos^2 \phi = 1 \tag{19}$$

Hyperboloid of 2 Sheets

$$-x^2 - y^2 + z^2 = 1 \quad \Rightarrow \quad -\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi = 1 \tag{20}$$

Cone

$$x^2 + y^2 = z^2 \quad \Rightarrow \quad \rho^2 \sin^2 \phi = \rho^2 \cos^2 \phi \ \text{ or } \ \tan \phi = 1 \ \textit{ so } \ \phi = \frac{\pi}{4} \tag{21}$$

Paraboloid

$$z = x^2 + y^2 \quad \Rightarrow \quad \rho \cos \phi = \rho^2 \sin^2 \phi \tag{22}$$

so we see that in cylindrical coordinates the

hyperboloid of 1 and 2 sheets, and the *paraboloid*

becomes easier, whereas in spherical polar coordinates the

sphere and the *cone*

becomes easier.

Triple Integrals

So now we ask, how do triple integrals

$$V = \iiint_V F(x, y, z) dV \tag{23}$$

change in each of these two coordinate systems and, in particular, in cylindrical and spherical coordinates.