Clique Domination in Interval Graphs

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Abstract - In this paper, we investigate the problem of dominating clique in interval graphs as interval graphs have wide range of applications in the field of scheduling and genetics. For this purpose, certain classes of interval graphs are taken into consideration and their dominating cliques are obtained.

Keywords - *Interval graph, Dominating set, Clique, Dominating clique*

I. INTRODUCTION

Graph Theory is one of the most thriving branches of mathematics having wide range of applications in many branches of Sciences, Engineering, Computer Sciences, etc. By a graph we mean a finite, simple, connected and undirected graph G (V, E), where V denotes its vertex set and E denotes its edge set.

Graphs presented in this article are all finite and connected Interval Graphs. Interval graphs are a special class of circular-arc graphs that can be represented with a set of arcs that do not cover the entire circle. The extensive study of interval graphs has been done for several decades by both mathematicians and computer scientists. Let $I = \{I_1, I_2, I_3, \ldots.., I_k\}$ be an interval family, where each I_i is an interval on the real line and $I_i = [a_i, b_i]$, for i = 1, 2, 3, …, k. Here a_i is called the left end point and b_i is called the right end point. Without loss of generality, one can assume that, all end points of the intervals are distinct numbers between 1 and 2k.. The intervals are named in the increasing order of their right end points. The graph G (V, E) is an interval graph if there is one-to-one correspondence between the vertex set V and the interval family I. Two vertices of G are joined by an edge if and only if their corresponding intervals in I intersect. That is if $I_i=[a_i, b_i]$ and $I_j=[a_j, b_j]$, then I_i and I_j will intersect if $a_i < b_j$ or $a_j < b_i$. Interval graphs are rich in combinatorial structures and have found applications in several disciplines such as traffic control, ecology, biology, computer sciences and particularly useful in cyclic scheduling and computers storage allocation problems etc.

The dominating set problem is the problem of determining whether a given graph G has a dominating set of a specific size k. A subset S of a vertex set V of a graph G is said to be a dominating set, if every vertex in V – S is adjacent to some vertex in S. The set S is a minimal dominating set, if the set S – { u } is not a dominating set for any u Є S. The domination number of G is the minimum number of vertices in a dominating set and is denoted by $\gamma(G)$. Dominating sets play a leading role in algorithms and in combinatorics. The topic has long been of interest to researchers [1], [2]. There are books that are entirely dedicated to dominating sets. Numerous variants of this dominating problem have been studied in [3] and [4]. Dominating set was one of the first problems recognized as NP complete.

The concept of clique domination in graphs is introduced by Cozzens and kelleher [5]. Study of Clique Domination in P_5 – free graphs is carried out by Basco [6] and bounds for clique domination number are obtained. A dominating set S is called a clique dominating set of G if < S > is complete. A graph with a dominating clique is called a clique dominating graph. The clique domination number $\gamma_{cl}(G)$ of G is the minimum cardinality of a dominating set clique. The minimum cardinality possible for a dominating clique is one and such cliques are called as central vertex or central point.

Clique dominating sets have a wide range of applications. In setting up the communication links in a network one might want a strong core group that can communicate with each other member of the core group and so that everyone outside the group could communicate with someone within the core group.

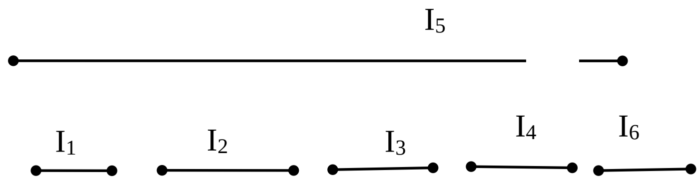
Throughout this paper, a complete graph with n vertices is denoted by K_n, a path graph on n vertices is denoted by P_n and a Cycle graph on n vertices is denoted by C_n.

2. Dominating Cliques in interval graphs

Theorem 2. 1: Let $I = \{I_1, I_2, \ldots\ldots\ldots, I_k\}$, $k \geq 2$ be an interval family corresponding to an interval graph G. Suppose that there exists an interval I_i Є I such that every other interval of the family can't but simply dominate any other interval except the interval I_i, then the interval graph G has a dominating clique of size one.

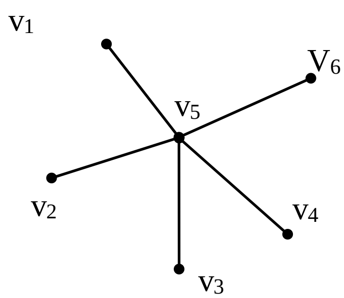
Proof: Let v_i be the vertex corresponding to the interval I_i respectively for i = 1, 2, 3, …, k. Let the interval family $I = \{I_1, I_2, \ldots\ldots\ldots, I_k\}$, $k \geq 2$ satisfy the condition mentioned in the theorem. As I_i dominates every interval of the family, the set S = { V_i } is the dominating set and moreover it is a minimum dominating set. The subgraph induced by a single vertex is always complete. The induced subgraph, < s > is a complete graph. Hence, in this case, the interval graph G has a dominating clique of size one.

Illustration 2. 1. 1: Let the interval family I = { I_1, I_2, I_3, I_4, I_5, I_6} corresponding to the interval graph G be asfollows:



Interval family I

The interval Graph G corresponding to the interval family I will be as follows



Interval Graph G

Clearly the interval family I satisfies the conditions mentioned in the theorem 4.1 for k=6 and i = 5. Therefore the interval family I has a dominating clique of size one. The dominating clique is the set { v_5 }.The minimum dominating clique is S = { v_5 }. The subgraph induced by the set S will be as follows:

The induced subgraph, < S >:

Theorem 2. 2: Let I = {I_1, I_2, ………, I_k}, $k \geq 2$ be an interval family analogous to an interval graph G. In a condition, wherein the intervals I_i, I_j are intersecting intervals and any interval of I other than I_i and I_j doesn't dominate any other interval except I_i or I_j, but not both, then

1. The interval graph G has dominating clique of size two, if some of the intervals in I-{ I_i, I_j } dominate I_i and some of the intervals in I-{ I_i, I_j } dominate I_j and

2. The interval graph G has dominating clique of size one , if all the intervals in I-{ I_i, I_j } dominate I_i or all the intervals in I-{ I_i, I_j } dominate I_j .

Proof: Let the interval family I = {I_1, I_2, ………, I_k}, $k \geq 2$ satisfy the conditions mentioned in the hypothesis. Here two cases may arise

Case (i): Let some of the intervals in I-{I_i, I_j} dominate I_i and the remaining intervals in I-{I_i, I_j} dominate I_j. Let v_1, v_2, v_3, ….., v_k be the vertices corresponding to the intervals I_1, I_2, ………, I_k respectively. By the conditions for domination between the intervals of the interval family I, it is clear that except the intervals I_i and I_j, the remaining k-2 intervals either dominate I_i or I_j. The set S = {I_i, I_j} is a dominating set. It implies, $\gamma(G) \leq 2$……….. (1).

Either the interval I_i or I_j does not dominate the entire set of intervals in I-{ I_i, I_j }So $\gamma(G) > 1$………..(2)

From (1) and (2), it is clear that, $\gamma(G) = 2$

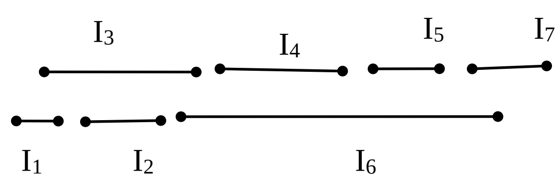
But the intervals I_i and I_j are mutually intersecting intervals. Therefore, the subgraph induced by the set S is complete. Hence the set S is a dominating clique with minimum cardinality.

Case (ii): Let all the intervals in I-{I_i, I_j} dominate only the interval I_i. Then all the intervals of I other than I_i will not dominate any other interval except I_i. Therefore, all the intervals of the interval family I except I_i dominate I_i. Implies, S = { v_i} is a minimum dominating set. Therefore

$\gamma(G) = 1$…………. (1)

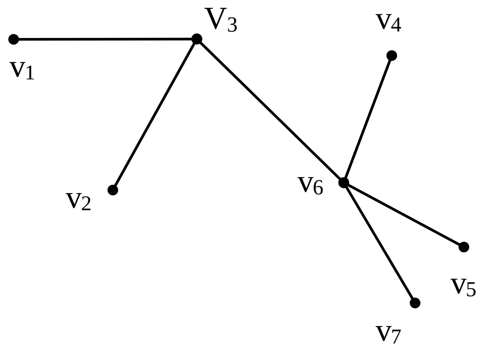
But every set with unique element always induces a complete graph. Hence, S = {v_i} is a minimum dominating clique.

Illustration 2. 2. 1: Let the interval family I = {I_1, I_2, I_3, …., I_7} corresponding to the interval graph G be as follows:



Interval family I

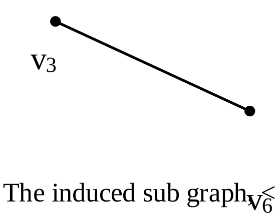
The interval Graph corresponding to the interval family I will be as follows



The interval Graph G

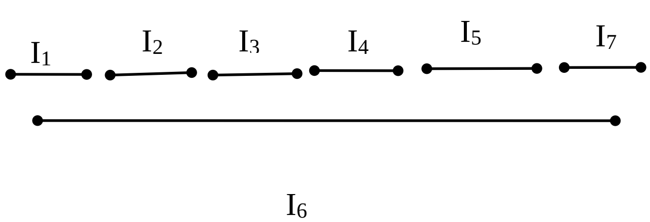
Here, the interval I_3 intersects the interval I_6 and any interval of I other than I_3 and I_6 does not dominate any other interval except I_3 or I_6, but not both. Moreover some of the intervals in I-{I_3, I_6} dominate I_3 and some of the intervals in I-{I_4, I_5} dominate I_6. It follows that conditions mentioned in the theorem 4. 2 are satisfied for i = 3 and j = 6 (case i). Hence, the set S = {V_3, V_6} is the minimum dominating clique.

The sub graph induced by the set S will be as follows:



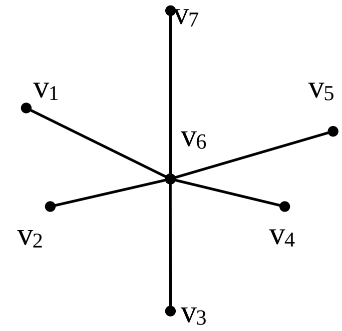
The induced sub graph, < S >:

Illustration 2. 2. 2: Let the interval family I = {I_1, I_2, I_3, …., I_7} corresponding to the interval graph G be as follows



Interval family I

The interval Graph corresponding to the interval family I will be as follows:



Interval Graph G

Here, the interval I_6 intersects the interval I_7 and any interval of I other than I_6 and I_7 does not dominate any other interval except I_6.. It follows that conditions mentioned in the theorem 4. 2 are satisfied for i = 6 and j = 7 (case ii). Hence, S = { v_6 } is the minimum dominating clique.

The sub graph induced by the set S will be as follows:

The induced sub graph, < S >:

Theorem 2.3: Let the interval family I = {I_1, I_2, ………, I_k}, k ≥ 2 corresponding to an interval graph G be such that every interval of the family dominates every other interval of the family. Then the interval graph G has a dominating clique of size one.

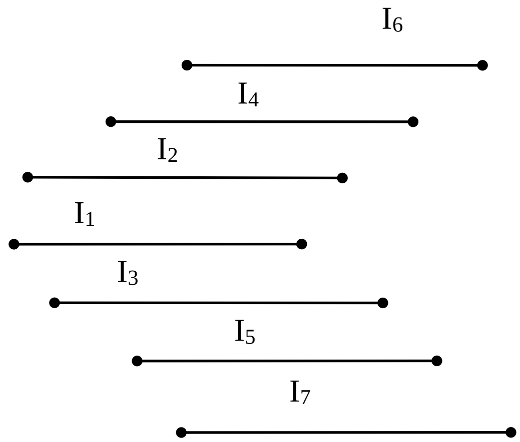
Proof: Let the interval family I = {I_1, I_2, ………, I_k}, k ≥ 4 satisfy the conditions mentioned in the hypothesis. Let v_1, v_2, v_3, ….., v_k be the vertices corresponding to the intervals I_1, I_2, ……, I_k . By the conditions for domination between the intervals of the interval family I, it is clear that

nbd{ v_i } = { 1, 2, …………….v_{i-1}, v_{i+1}, ….k }

For i = 1, 2, …………………., k.

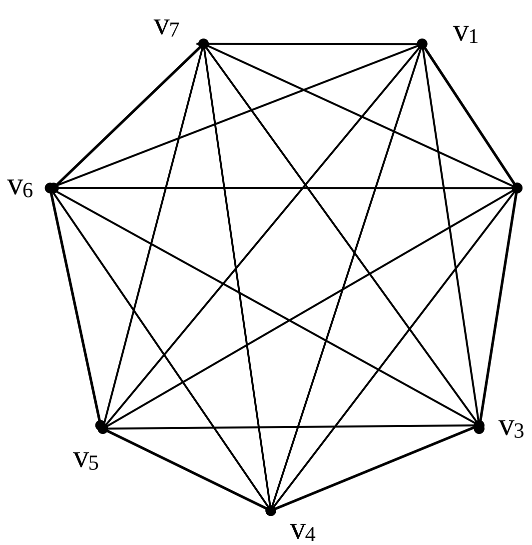
Every vertex is a dominating set and moreover every vertex induces a complete subgraph. Therefore the set S = { v_i } for I = 1, 2, ….., k is a dominating clique. Hence in this case, the interval graph G has a dominating clique of size one**.**

Illustration 2. 3. 1: Let the interval family I = {I_1, I_2, I_3, …., I_7} corresponding to the interval graph G be as below. Here every interval of the family intersects every other member of the family.



Interval family I

The interval Graph corresponding to the interval family I will be as follows



The interval Graph G

It follows that conditions mentioned in the theorem 2. 3 are satisfied for i = 1, 2, …., 7. Hence, S = $\{v_i\}$ for i = 1, 2, ……, 7 are all dominating cliques.

Theorem 2. 4: Let I = $\{I_1, I_2, \ldots\ldots\ldots, I_k\}$, $k \geq 1$ be an interval family corresponding to an interval graph G. If the interval family I has no subinterval family I'= $\{ I_p, I_q, I_r, I_s, I_t\}$, where $1 \leq p < q < r < s < t \leq k$; such that for any three consecutive intervals in I', the middle interval does not dominate any other interval of I' except the intervals that precede it and succeed it, then the graph G has a dominating clique.

Proof: Let us first prove the result for k=1. In this case, I = $\{ I_1 \}$. Therefore there does not exist a sub interval family I' of the interval family I as stated in the theorem. Moreover, every trivial graph has a dominating clique. The result is true for k = 1. Suppose that the result is true for k = n. Now we shall prove the result for k = n + 1.

Let I' = $\{ I_p, I_q, I_r, I_s, I_t\}$, where$1 \leq p < q < r < s < t \leq k$; such that for any three consecutive intervals in I', the middle interval does not dominate any other interval of I' except the intervals that precede it and succeed it.

It follows that,
I_q dominates I_p and I_r;
I_r dominates I_q and I_s and
I_s dominates I_r and I_t.

Except these dominations, no other dominations can take place in the interval family I'. In such cases, the interval graph G' corresponding to the interval family I' is a path graph. Let the interval family I has no such sub interval family I' satisfying the conditions stated in the theorem. Implies that the interval graph G has no induced path graph P_5. By [7], there exists a vertex of G, say v that is not a cut point. The graph G – v is a connected graph of order n without any induced path graph P_5. As a result, by our assumption, G – v has a dominating clique. Let S be a dominating clique of the interval graph G – v. Here two cases will arise.

Case (i): The vertex v may be adjacent to one of the vertices of the vertex set S. Then, S is a dominating set of the graph G and the subgraph induced by S is complete. In such cases, the set S itself is the dominating clique.

Case (ii): The vertex v may not be adjacent to any vertex of S. As G is a connected graph, the vertex v must be adjacent to some other vertex 'u' in G. Then S' = $\{ u \} \cup (N(u) \cap S)$ is a dominating clique of G [5]. But, the converse of the theorem is not true. We shall prove this with the help of an example.

Let the interval family I = $\{I_1, I_2, I_3, \ldots, I_{10}\}$ corresponding to the interval graph G be as follows

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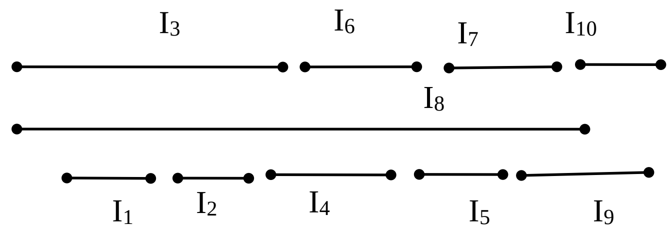
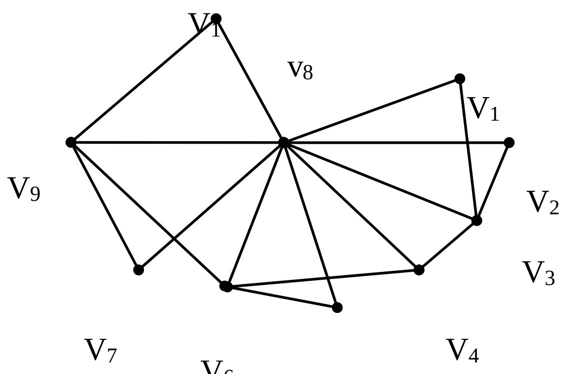


Interval family I

The corresponding interval graph G will be as follows:



Interval Graph G

In this case the interval graph G has a dominating clique of size one and an induced P_5. Moreover the minimum dominating clique is { v_5 }.

II. ACKNOWLEDGMENT

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III. REFERENCES

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