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Research Article

On the significance of the Hahn-Banach theorem based on its historical background, formulation, implications, applications, and contributions

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Abstract

It is known that without the Hahn-Banach theorem, functional analysis would be very different from the structure we know today. Among other things, it has proved to be very appropriate form of the axiom of choice for analysts. In its elegance and power, the Hahn-Banach theorem is a favorite of almost every analyst. Its principal formulations are as a dominated extension theorem and as a separation theorem. In this paper we give an overview of the significance of the Hahn-Banach theorem to Analysis based on its historical background, formulation, implications, applications, significance and importance to Analysis and its contributions after discovery.

Keywords: Formulation of the theorem; Extension of linear functionals; Riesz's contributions; Helly's contributions; Hahn and Banach's contributions; F.Murray contributions.

Introduction

Without the Hahn-Banach theorem, functional analysis would be very different from the structure we know today. Among other things, it has proved to be very appropriate form of the axiom of choice for analysts. In its elegance and power, the Hahn-Banach theorem is a favorite of almost every analyst. Its principal formulations are as a dominated extension theorem and as a separation theorem. The condition of completeness required by Hahn and Banach to develop the theorem is provided in [1-3] while [4] offers what Banach later did in his book without this condition of completeness. Murray in [5] brought an interesting contribution to Hahn-Banach theorem. He proved the complex version of Hahn-Banach theorem. His method was perfectly general and was used in [6] by Bohnemblust and Sobyzk who proved it for arbitrary complex normed spaces and were the first to refer to the theorem as the Hahn-Banach theorem incidentally.

The significance of Hahn-Banach theorem to Analysis is of interest to our work [7-15]. The study involves also singling out the contributions of Hahn-Banach theorem after discovery. A basic knowledge on the problems that lead to the discovery of Hahn-Banach theorem we are studying today is fundamental to our study. In [16,17], we find the problems Riesz worked on in trying to develop Hahn-Banach theorem. In [18,19] we find the contributions of Helly's perspective of viewing things in terms of extending continuous linear forms which gave the precursor to the argument that Hahn and Banach later used. In [20-21], we find Hahn and Banach's contributions in the proof of the theorem by reducing the problem to showing that a continuous linear form defined on a subspace M of a normed space can be extended to an enlargement by one vector to $[M\cup\{x\}]$ without increasing its norm while [22] offers a concept of Minkowski's earlier ideas concerning convexity which Helly used to link his general norm.

Formulation of the theorem

Hahn-Banach theorem (Extension of linear functionals)

Let X be a real vector space and P a sub linear functional on X. Furthernore, let f be a linear functional which is defined on a subspace Z of X and satisfies the condition

$$f(x) \leq P(x) \forall x \in Z \dots\dots\dots\dots\dots\dots (1)$$

Then f has a linear extension $\hat{f}$ from Z to X satisfying

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$\hat{f}(x) \leq P(x) \forall x \in X$ ……………………….(2)

That is $\hat{f}$ is a linear functional on X, satisfies (2) on X and $\hat{f}(x) = f(x) \forall x \in Z$

Hahn-Banach theorem (Generalised)

Let X be a real or complex vector space and P a real-valued functional on X which is subadditive, that is $\forall\ x, y \in X$ in the trivial case $Z = \{0\}$

Hahn-Banach theorem (complex version)

Let X be a complex vector space,M a subspace of X and let P be a seminorm on X i.e $P: X \to \mathbb{R}$. Suppose that f is a complex valued linear functional on M such that

$|f(x)| \leq P(x) \forall x \in M$ then $\exists$ a linear extension g *of* f such that $|g(x)| \leq P(x) \forall x \in X$

Implication of Hahn-Banach theorem

The Hahn-Banach theorem implies that for any non-zero vector x, there is a continuous linear functional f on X such that $f(x) = P(x) \neq 0$. Consequently, if every continuous linear functional vanishes on a vector x then $x = 0$.

Historical background of Hahn-Banach theorem

Hahn and Banach independently proved the theorem for the real case [1-3]. Then there was Murray's extension to the complex case [5], (Once you realize that $f(x) = Ref(x) - i\ Ref(ix)$, can continuous linear maps be extended as easily as linear functionals? Banach and Mazur had already proved that they could not but it was not until Nachbin's result that a definitive answer was achieved to this more general question.

Riesz in [16, 17] and Helly in [18] obtained the forerunners of the theorem in the turbulent mathematical world of the early 1900's.

Problems that lead to development of Hahn-Banach theorem:

Riesz's Contributions

Borrowing some concepts already done in Hilbert spaces, Riesz's in [16-17] set out to solve the following problem: for $P > 1$ (So he could use the Holder and Minkowski inequalities which he had just generalized.

Problem1

Given infinitely many y_s in $L_q[a,b]$ and scalars C_s find x in $L_p[a,b]$ such that

$\int_a^b x(t)\, y_s(t) dt = C_s$

Remark:

i. His solution and method of attack bore no resemblance to what had come before.
ii. For there to be such an x, he showed that the following necessary and sufficient connection between the $y's$ *and the* $C's$ had to prevail:
For any finite set of indices S and any scalars a_s there should exist $k > 0$ such that

$$\left|\sum a_s C_s\right| \leq k \left(\int_a^b \left|\sum a_s y_s\right|^q\right)^{\frac{1}{q}}$$

This implies that if $\sum a_s y_s = 0$, then $\sum a_s c_s = 0$ as well. So if we define a linear functional f on the linear span M of the $y's$ in $L_q[a,b]$ by taking $f(y_s) = C_s$, the f
So obtained is well defined. Not only that , for any y in M,
$|f(y)| \leq k\, \|y\|_q$
So in todays language we would say that f is bounded or continuous on M
iii. If there is an x in L_p which solves problem 1, then he showed that f has a continuous extension to the whole space. The ability to solve linear equations, in other words, implies being able to continuously extend a bounded linear functional to the whole space. Thus Riesz's solution to Problem 1 constitutes a special case of the Hahn-Banach theorem.
iv. Riesz then changed spaces and turned to the following variant of the problem.

Problem 2

Given $y_s \in C_{[a,b]}$ and scalars C_s find $x \in BV[a,b](bounded\ variation)$ such that $\int_a^b y_s(t) dx(t) = C_s$.

Adapting his earlier methods, he solved it with a condition that looked very much like the boundedness condition. He realized the importance of the condition and proved that any continuous additive map satisfied such a condition and conversely, where by continuous he meant sequently continuous with respect to the sup norm. In each case he proved a special case of the Hahn-Banach theorem and identified the continuous dual of a normed space.

Helly's Contributions

Riesz did not view things in terms of defining and extending continuous linear forms. In [18], Helly viewed things in terms of extending continuous linear forms and gave the precursor to the argument that Hahn in [20] and Banach [21] each used later to prove the Hahn-Banach theorem-namely, by reducing the problem to showing that a continuous linear form defined on a sub space M of a normed space can be extended to an enlargement by one vector to $[M \cup \{x\}]$ without increasing its norm. Helly then set to solve the following problem:

Problem 3

Given $u_i \in X', (C_i) \in C^N$, find $x \in X: \langle x, u_i \rangle = C_i \forall i \in N$.

He split the problem into two parts:

i) Find a linear map $f: X' \to C$ such that $|f(u)| \leq k\|u\| \ \forall k > 0$ and all $u \in X'$ with $f(u_i) = C_i$ and

ii) Once f has been found (if it can be found), find $x \in X: \langle x, u \rangle = f(u) \forall u \in X'$

Helly solved (i) by induction and a result of his on convex sets; he discovered that the x of (ii) could not always be found. He (and Riesz) thus became the first to exhibit non-reflexive Banach spaces.

Remark

Helly's principal contributions to Hahn-Banach theorem were as follows:

i. He defined and worked with a general sequence space endowed with a general norm.
ii. He utilized various notions about convexity
iii. He introduced the rudiments of duality theorem
iv. He realized the generality of Riesz's continuity condition and defined the infimum of the k that satisfies the condition as the Maximalzahl, i.e what is now called the norm of the linear functional.

Hahn and Banach's contributions

Hahn and Banach went a long way to shaping functional analysis as we know it today. They:

i. Defined the general normed space. Banach in [1] and [2] and Hahn in [3] did it independently. Each of them required completeness. Banach later in [6] removed it in his book, distinguishing between normed and Banach spaces (the general notion of norm was in the air at this time.
ii. Abandoned systems of linear equations and considered the general problem of extending a continuous linear form defined on a general normed space, not a sequence as Helly had done. Thus they formulated the theorem as we know it today.
iii. Defined the dual space of a general complete normed space and proved that it too is a complete normed space with respect to the standard norm.
iv. Defined reflexivity and realized that a normed space X is generally embedded in its second dualX''
v. They used transfinite induction (Helly had used ordinary induction) the way it was used here became an essential tool of the analyst from that time forward.

Immediate consequences of the work of Hahn and Banach

i. Norm-Preserving extensions:-Given a continuous linear functional f defined on a subspace of a normed space, there exists a continuous linear extension F defined on the whole space such that $\|f\| = \|F\|$.
ii. Non trivial continuous linear forms:- A linear form f on a locally convex space X is continuous iff there is a continuous seminorm P on X such that $|f| \leq P$. Moreover if X is Hausdorff and $x \neq 0$, there must be a continuous seminorm $P \ on \ X: P(x) \neq 0$.
iii. Sehie in [23] added some more consequences of equivalents of Hahn-

Banach theorem that belong to a partial realm.

Theorem (Hahn-Banach theorem) [23, theorem 2.1]

Let the real function P on the real linear space X satisfying P(x+y) ≤ P(x)+P(y) ; P(αx) = $\alpha P(x)$; $\alpha \geq 0; x, y \in X$. Let f be a real linear functional on a subspace Y of X with f(x) ≤ P(x) ; $x \in Y$. Then there is a real linear functional F on X for which F(x)=f(x); $x \in Y$; F(x) ≤ P(x), $x \in X$.

Significance of Hahn-Banach theorem to Analysis

The Hahn-Banach theorem is a powerful existence theorem whose form is particularly appropriate to applications in linear problems. Some of the ways in which it resonates throughout functional analysis include:

i. Duality theorem: The dual space X' of a normed space X is a Banach space(whether or not X is)
ii. Banach's solution of the 'easy' problem of measure [7]
iii. Applications to convex programming [8]
iv. A formulation of thermodynamics [9]
v. Proof of the existence of Green's functions [10]
vi. Applications to game theory [11]
vii. Applications to control theory [12, 15]
viii. Helly's criterion for solving systems of linear equations in reflexive normed spaces [14]
ix. Cauchy integral theorem for vector valued analytic functions $x: D \to X, X$ a Banach space, D a domain of the complex plane $\mathbb{C}$[14]
x. In [24], Horrath gave generalized abstract of the Von Neumann-Sion minimax theorem and reduced to Sion's minimax theorem based on Hahn-Banach theorem.
xi. In [25], Ben-El- Mechaiekh presented a large number of examples of the Hahn-Banach type or of the KMM type and showed that the Hahn-Banach theorem is of the KMM type.
xii. In [26], Applications of some elementary principles of convex analysis were introduced. A geometric approach in the theory of minimax inequalities, which has numerous applications in different areas of mathematics was presented.

Contributions of Hahn-Banach theorem after discovery

The Hahn-Banach family of theorems more aptly describes what exists today and it is a thriving mathematical enterprise. To name just a few recent developments, we have:

1. Nyman in [27] chose a particular form of Hahn-Banach theorem and Fan's lemma and called it Fan's N+1 lemma and showed that the N+1 lemma is equivalent to Borsuk-Ulam theorem and directly implies the weak Sperner lemma.
2. Park in [28] indicated that Hahn-Banach theorem implies a mimimax theorem of Fan; it also implies Fan's convex inequality theorem and also Markov-Kakutani theorem.
3. Burgin in [29] using nonstandard analysis, an analog of the Hahn-Banach theorem for hyperfunctions is obtained.
4. Ding in [30] some conditions for anon-locally convex space such as the $l_{p, 0<p<1}$ to have the Hahn-Banach extension property.
5. Plewnia in [31] instead of a linear subspace of real space X, let C be a non-empty convex subset of X.Let $P: X \to \mathbb{R}$ be a convex function and let Let $f: C \to \mathbb{R}$ be a concave function with Let $f(x) \leq P(x)$ on Let C. Then there exists a linear function $g: X \to \mathbb{R}$ and a real constant Let a such that Let $g(x) + a \leq P(x) \forall x \in X$ and $f(x) \leq P(x) + a \; \forall \; x \in C$.
6. Ruan in [32] A Hahn-Banach theorem for bisublinear functionals.
7. Sorjoem in [33] A Hahn-Banach theorem in linear orthogonality spaces, left vector spaces over a division ring with an abstract orthogonality relation.

Su in [34] A Hahn-Banach theorem for a class of linear functionals on probabilistic normed spaces.

Conclusions

Indeed, without the Hahn-Banach theorem, functional analysis would be very different from the structure we know today. In this paper, a summary of the way Hahn-Banach theorem was developed, applications, implications and how it has contributed to mathematical analysis has been illustrated. This gives us the significance of this theorem to analysis. We have also seen the contributions of various mathematicians in building up Hahn-Banach theorem.

Conflict of interest

The authors declare no conflict of interests.

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