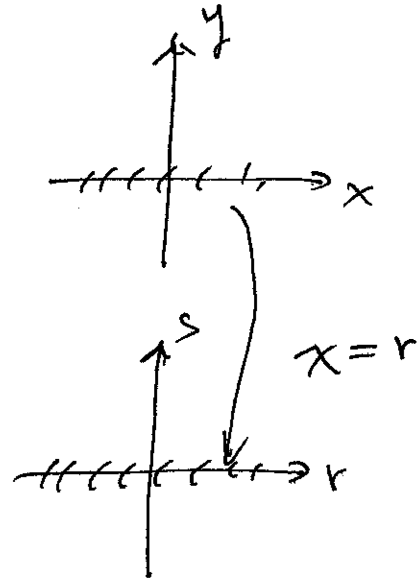
Math 4315 — PDEs

More examples w/ B.C.

<u>Ex!</u> $u_x + 2x\,u_y = 2x \qquad u(x,0) = 2x^2$



so $\quad u_s = u_x x_s + u_y y_s$

$$\begin{array}{lll} & & s=0 \\ \text{choose} & x_s = 1 & x=r \\ & y_s = 2x & y=0 \\ \text{so} & u_s = 2x & u = 2r^2 \end{array}$$

so $\quad x = s + a(r) \qquad s=0 \quad x=r \Rightarrow a=r \quad$ so $\quad x = s+r$

$y_s = 2x = 2s + 2a(r) \qquad y = s^2 + 2a(r)s + b(r)$

$\qquad\qquad\qquad\qquad\qquad\quad = s^2 + 2rs + b(r)$

$=$

$s=0 \quad y=0 \quad$ so $\quad b(r)=0 \quad$ so $\quad y = s^2 + 2rs$

$u_s = 2x = 2(s+r) \qquad u = s^2 + 2rs + c(r)$

$s=0 \quad u = 2r^2 \quad$ so $\quad c(r) = 2r^2 \quad$ so $\quad u = s^2 + 2rs + 2r^2$

so $x = s + r$ $\qquad$ $x^2 = s^2 + 2rs + r^2$

$y = s^2 + 2rs$ $\qquad$ $y = s^2 + 2rs$

$u = s^2 + 2rs + 2r^2$ $\qquad$ $x^2 - y = r^2$

$$u = s^2 + 2rs + r^2 + r^2$$

$$= (s+r)^2 + r^2 = x^2 + x^2 - y = 2x^2 - y$$

sol^n $u = 2x^2 - y$

check $u(x,0) = 2x^2$ ✓

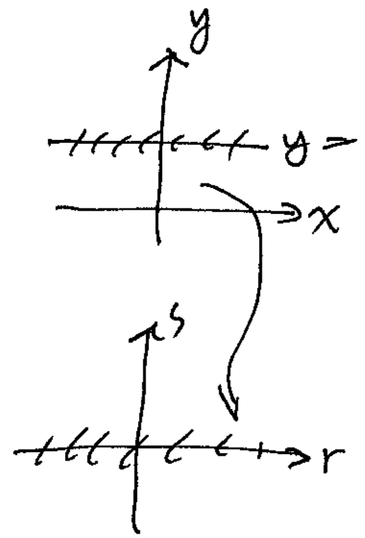
$u_x = 4x, \quad u_y = -1$

so $u_x + 2xu_y \overset{?}{=} 2x$

$4x + 2x(-1) \overset{?}{=} 2x$

$2x \overset{?}{=} 2x$ ✓

Ex 2 $\quad x u_x - y u_y = x \qquad u(x,1) = 0$



$x_s = x \qquad s = 0$

$y_s = -y \qquad x = r$

$u_s = x \qquad y = 1$

$\qquad\qquad u = 0$

$x_s = x$ so $x = a(r)e^{s} \quad s=0 \ x=r \Rightarrow \ r = a(r)e^{0}$
$\qquad = a(r)$

So $\boxed{x = re^{s}}$

$y_s = -y$ so $y = b(r)e^{-s} \quad s=0, y=1 \Rightarrow b(r)=1$

So $\boxed{y = e^{-s}}$

$u_s = x = re^{s}$ so $u = re^{s} + C(r) \quad s=0 \ u=0$

$\Rightarrow \quad 0 = r + C(r) \Rightarrow C(r) = -r$

so $\boxed{u = re^{s} - r}$ $\rightarrow$ $\boxed{u = x - xy \quad \text{Sol}^{n}}$

Now $\quad xy = re^{s} e^{-s}$
$\qquad = r$

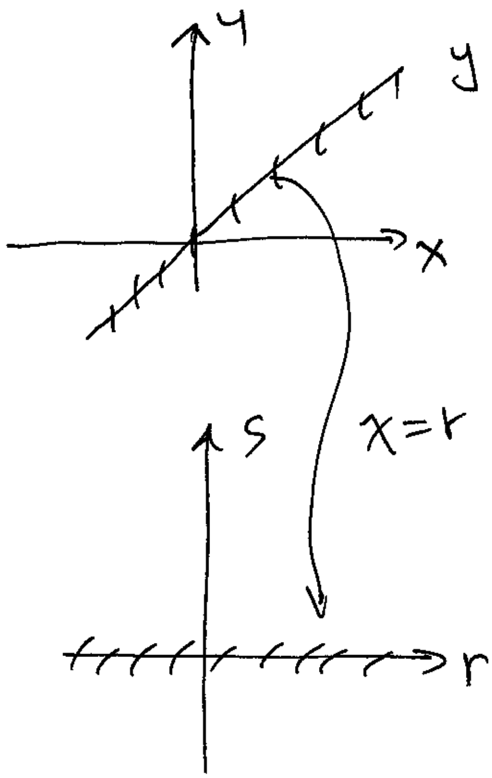
Ex 3 $\quad u u_x + y u_y = u + 2y^2 \quad u(x,x) = x^2 + x$

$u_s = u_x x_s + u_y y_s$ (again)

$$\left.\begin{aligned} x_s &= u \\ y_s &= y \\ u_s &= u + 2y^2 \end{aligned}\right\} \text{ we did this last Wed}$$

Boundary

$y = x$ where $u = x^2 + x$



So ~~th~~ new BC

$s = 0$

$x = r$

$y = x = r$

$u = x^2 + x = r^2 + r$

$y_s = y \Rightarrow y = b(r)e^s \quad s=0 \; y=r \Rightarrow b(r)=r$

$$\boxed{\text{So } \; y = re^s}$$

Next $\quad u_s = u - 2y^2 \quad$ or $\quad u_s - u = 2r^2e^{2s}$

$\mu = e^{-s}$ so $\quad \frac{\partial}{\partial s} e^{-s}u = 2r^2e^s$

so $\quad e^{-s}u = 2r^2e^s + c(r)$

$s=0 \quad u = r^2+r \Rightarrow \quad e^0 r^2+r = 2r^2e^0 + c \qquad c(r) = r - r^2$

So $\quad e^{-s}u = 2r^2e^s + r - r^2 \Rightarrow \quad \boxed{u = 2r^2e^{2s} + (r-r^2)e^s}$

$x_s = u = 2r^2e^{2s} + (r-r^2)e^s \qquad x = r$ when $s=0$

$x = r^2e^{2s} + (r-r^2)e^s + a(r) \qquad r = r^2 + (r-r^2) + a(r)$

$\Rightarrow a(r) = 0$

$$\boxed{x = r^2e^{2s} + re^s - r^2e^s}$$

$u - x = 2r^2e^{2s} + \cancel{(r-r^2)}e^s - r^2e^{2s} - \cancel{(r-r^2)}e^s$

$= r^2e^{2s} = y^2 \qquad \boxed{\text{sol } \; u = x + y^2}$