Math 6345 - AODE's

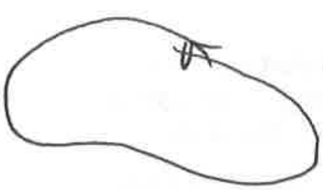
Thm - A closed orbit must enclose at least 1 critical pt

Proof: By contradiction

Suppose that the closed orbit C encloses some region R with no C.P. so if

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y)$$

then $f^2 + g^2 \neq 0$ in R



so $\int_C d\theta = 2\pi$

Assume that $f \neq 0$ then

$$\tan\theta = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{g}{f}$$

then $\sec^2\theta \, d\theta = \dfrac{f\,dg - g\,df}{f^2}$

$$d\theta = \frac{f\,dg - g\,df}{f^2} \cdot \frac{1}{1+\tan^2\theta}$$

$$d\theta = \frac{f\,dg - g\,df}{f^2} \cdot \frac{1}{1+\frac{g^2}{f^2}}$$

$$= \frac{f\,dg - g\,df}{f^2+g^2}$$

$$= \frac{f(g_x\,dx + g_y\,dy) - g(f_x\,dx + f_y\,dy)}{f^2+g^2}$$

$$= \frac{fg_x - gf_x}{f^2+g^2}\,dx + \frac{fg_y - gf_y}{f^2+g^2}\,dy$$

so $$\int_c d\theta = \int_c \frac{fg_x - gf_x}{f^2+g^2}\,dx + \frac{fg_y - gf_y}{f^2+g^2}\,dy$$

let $$P = \frac{fg_x - gf_x}{f^2+g^2} \qquad Q = \frac{fg_y - gf_y}{f^2+g^2}$$

$$= \int_c P\,dx + Q\,dy$$

$$= \iint_R (Q_x - P_y)\,dA = 0$$

with P & Q chosen as these.

$$\neq 2\pi$$

Contradiction

Poincaré - Bendixson Thm

Suppose

(i) R is a closed, bounded region in $\mathbb{R}^2$

(ii) $f(x,y),\ g(x,y) \in C^1(\mathbb{R}^2)$

(iii) R does not contain a fixed pt

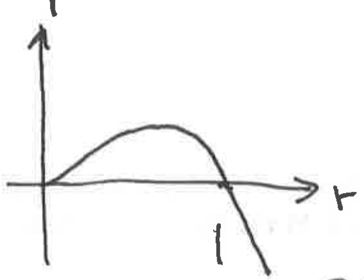
(iv) There is a curve C that is confined in R

- it starts in R & stays in R

Then R contains a closed orbit

Ex $\quad \dot{r} = r(1-r^2) + \mu r \cos\theta, \quad \dot{\theta} = 1, \quad \mu > 0$

when $\mu = 0$

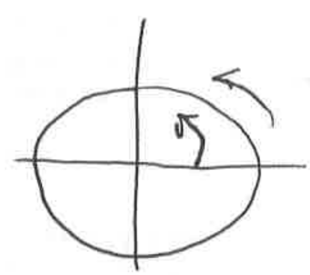


we seen $\dot{r} > 0$ when ~~r>0~~ $0 < r < 1$

& $\dot{r} < 0$ when $r > 1$

so $r = 1$ is

a stable limit cycle

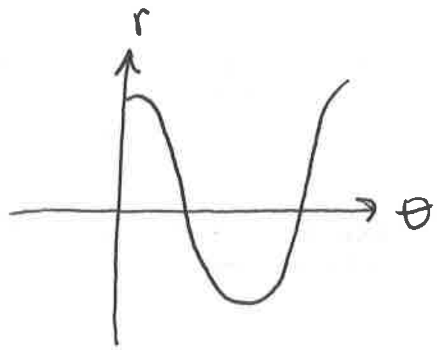


what about when $\mu \neq 0$.

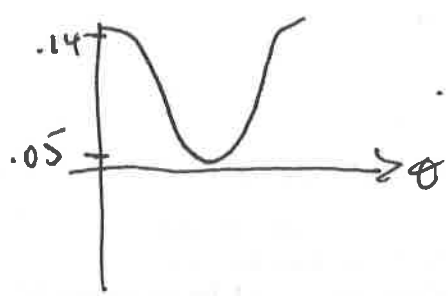
Say $\mu = 1/2$ for what values of r is

$\dot{r} > 0$ $\qquad$ $\dot{r} < 0$

$r = 1$

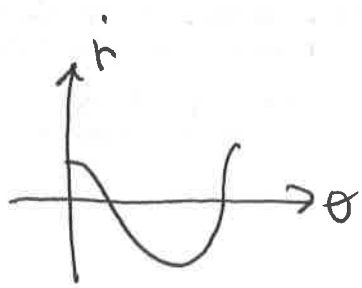


$r = .1$

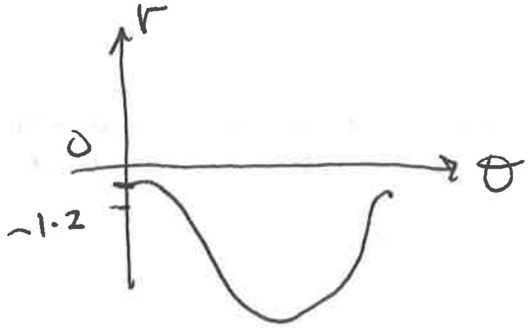


so $\dot{r} > 0$

$r = 1.2$

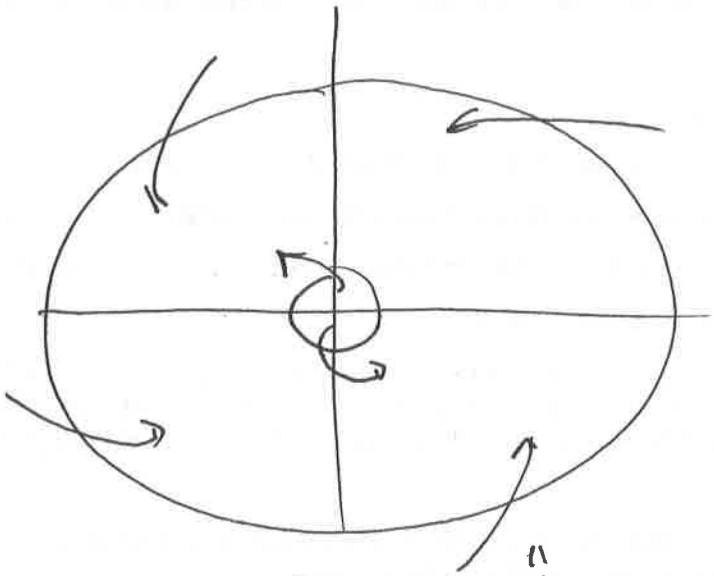


$r = 1.5$



$\dot{r} < 0$

So we have



So in essence we have a "trapping region".

How about for more general μ

if $$-1 \le \cos\theta \le 1$$

$$-\mu r \le \mu r\cos\theta \le \mu r$$

$$r(1-r^2) - \mu r \le r(1-r^2) + \mu r\cos\theta \le r(1-r^2) + \mu r$$

if $0 < \mu < 1$ $\quad \dot{r} < 0$ if $1 - r^2 + \mu < 0$

$$\Rightarrow \quad r < \sqrt{1+\mu}$$

&, $\dot{r} > 0$ if $1 - r^2 - \mu > 0 \Rightarrow r > \sqrt{1-\mu}$

trapping region $\quad \sqrt{1-\mu} < r < \sqrt{1+\mu}$