Math 4315 - PDE's

Regular Hyperbolic SF

Solve $a r_x^2 + b r_x r_y + c r_y^2 = 0$

and $$\frac{r_x}{r_y} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and follow the $+/-$

we will hit $u_{ss} - u_{rr} + \text{lots} = 0$

Elliptic SF

$$u_{ss} + u_{rr} + \text{lots} = 0$$

Note: Hyp SF $\longrightarrow$ Elliptic SF

$$u_{ss} - u_{rr} + \text{lots} = 0 \qquad u_{ss} + i^2 u_{rr} + \text{lots}$$

so we now follow the $\pm i$

The following 2 examples illustrate

Ex 1 $\quad u_{xx} + 2u_{xy} + 5u_{yy} = 0$

$b^2 - 4ac = 4 - 20 = -16 < 0$ so ellipti

so $\quad r_x^2 + 2r_x r_y + 5r_y^2 = 0$

$$\frac{r_x}{r_y} + 2\frac{r_x}{r_y} + 5 = 0$$

$$\frac{r_x}{r_y} = -\frac{2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

so $\quad r_x = (-1 \pm 2i) r_y \Rightarrow r_x - (-1 \pm 2i) r_y = 0$

$$\frac{dx}{1} = -\frac{dy}{-1 \pm 2i} \; ; \; dr = 0 \Rightarrow (-1 \pm 2i)x + y = c_1, \quad c_2 = r$$

$$r = R((-1 \pm 2i)x + y) = R(-x + y \pm 2ix)$$

pick $\quad r = -x + y, \quad s = 2x$

$u_{xx} = U_{rr} + 4U_{rs} + 4U_{ss}$

$u_{xy} = -1U_{rr} + 2U_{rs}$

$u_{yy} = U_{rr}$

sub $\quad U_{rr} + 4U_{rs} + 4U_{ss}$

$\quad - 2U_{rr} + 4U_{rs}$

$\quad + 5U_{rr} \quad = 0$

$\Rightarrow 4U_{rr} + 4U_{ss} = 0$

$\Rightarrow U_{rr} + U_{ss} = 0$

Ex2 $y^2 u_{xx} - 4y\, u_{xy} + 5 u_{yy} = 0$

$$y^2 r_x^2 - 4y\, r_x r_y + 5 r_y^2 = 0$$

$$\frac{r_x}{r_y} = \frac{4y \pm \sqrt{16y^2 - 20y^2}}{2y^2} = \frac{4y \pm \sqrt{-4y^2}}{2y^2}$$

$$= \frac{4y \pm 2iy}{2y^2} = \frac{(2 \pm i)}{y}$$

so $y\, r_x = + (2 \pm i) r_y \Rightarrow y\, r_x - (2 \pm i) r_y = 0$

$$\frac{dx}{y} = -\frac{dy}{(2 \pm i)} \; ; \; dr = 0 \qquad (2 \pm i)x + \frac{y^2}{2} = c_1, \; c_2 = r$$

$$r = R\left((2 \pm i)x + y^2/2 \right) = R\left(2x + y^2/2 \pm ix \right)$$

pick $r = x$, $s = 2x + y^2/2$

$r_x = 1$, $r_y = 0$, $r_{xx} = r_{xy} = r_{yy} = 0$

$s_x = 2$ $s_y = y$ $s_{xx} = s_{xy} = 0$ $s_{yy} = 1$

Chain Rule

$U_{xx} = U_{rr} + 4U_{rs} + 4U_{ss}$

$U_{xy} = 0\,U_{rr} + y\,U_{rs} + 2y\,U_{ss}$

$U_{yy} = 0\,U_{rr} + 0\,U_{rs} + y^2 U_{ss} + U_s$

Sub $y^2(U_{rr} + 4\cancel{U_{rs}} + 4U_{ss})$

$-4y(\quad y\cancel{U_{rs}} + 2y\,U_{ss})$

$+5(\quad y^2 U_{ss} + U_s) = 0$

$y^2 U_{rr} + (4-8+5)\,y^2 U_{ss} + 5U_s = 0$

$y^2 U_{rr} + y^2 U_{ss} + 5U_s = 0$

$$U_{rr} + U_{ss} + \frac{5U_s}{y^2} = 0$$

Now $r = x$

$s = 2x + y^2/2$

$y^2 = 2s - 4x$

$= 2s - 4r$

$$U_{rr} + U_{ss} + \frac{5U_s}{2s-4r} = 0$$

SF Elliptic