Toronto Math Circles: Junior
Third Annual Christmas Mathematics Competition
Saturday, December 17, 2016
1:00 pm - 3:00 pm

Each problem is graded on a basis of 0 to 10 points. All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Some partial credit may be given, but only when a contestant has shown significant and substantial progress toward a solution. Calculators are not allowed.

1. Eddy is trying to guess a four letter pass code. He makes the following five attempts

$$6087, 5173, 1358, 3825, 2531$$

   In each of his guesses, exactly two of the digits are in the correct pass code and these two correct digits are never in correct position. For example, if 1234 is a guess and 1 and 2 are in the correct pass code then 1 cannot be in the first position and 2 cannot be in the second position. Determine the correct pass code.

2. Using only a compass and a straightedge, describe a procedure to construct an 120° angle. Be sure to explain why this angle is 120°. Be sure to include a clearly labeled sketch.

3. Determine if the number 1,116,428,043 is a perfect square.

4. In a class, there are 10 students. A teacher randomly chooses 5 students and writes their names on a list. If exactly one of the student's name is John, what is the probability that John is on this list?

5. There are n cards placed faced down in a line. A move consists of flipping a faced down card to the faced up position and also flipping the immediate card to its right. Explain why after a finite number of moves there will be no more moves to be made.

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1. Denote $\lfloor x \rfloor$ to be the largest integer not greater than x. Let $\{x\} = x - \lfloor x \rfloor$. Find all triplets (x, y, z) that satisfy the system of equations

$$\begin{cases} x + \lfloor y \rfloor + \{z\} = 1.1 \\ \{x\} + y + \lfloor z \rfloor = 2.2 \\ \lfloor x \rfloor + \{y\} + z = 3.3 \end{cases}$$

2. There are n cards placed faced down in a line. A move consists of flipping a faced down card to the faced up position and also flipping the immediate card to its right. Show that after a finite number of moves there will be no more moves to be made.

3. Let x be a real number such that $0 < x < \frac{\pi}{4}$. Arrange the following four numbers in ascending order.

$$(\cos x)^{(\sin x)^{\sin x}}, (\sin x)^{(\cos x)^{\sin x}}, (\cos x)^{(\sin x)^{\cos x}}, (\sin x)^{(\sin x)^{\sin x}}$$

4. Let $\triangle ABC$ be an equilateral triangle inscribed in a circle. Let M be a point on the minor arc BC. Prove that $MA = MB + MC$. Be sure to include a clearly labeled sketch.

5. A positive integer p is called a Twin Prime Pair Base (TPPB) if p and $p + 2$ are both prime numbers. The Twin Prime Conjecture states that there are infinitely many values of p that are TPPB. For this problem, assume that this conjecture is true.
Let n be a positive integer. Denote $a_n = b_{n+1} + 1$ where b_n is the n^{th} smallest TPPB. Consider the polynomial
$$f(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_2x^2 + a_1x + a_0$$
Determine if this polynomial can be factored into a product of two non-constant integer coefficient polynomials.