A Case Study of Portfolio Optimization: Efficient Frontier

Maggie Foley* **Barry Thornton** **Biqing Huang** **Xiaowei Liu**

Abstract

In this study, we provide a template in Excel based on Matrix for the efficient frontier. Since the optimal portfolios usually have heavy concentrated stocks, we offer a solution on how to deal with heavy concentrated stocks.

Introduction

Modern portfolio theory is one of the common subjects introduced to students taking finance courses. Students will learn about the efficient frontier of portfolios made up of the set of efficient portfolios that offer the lowest risk for any given rate of return and the best return possible for each level of risk. Any other portfolio will be characterized as suboptimal, since it offers a higher level of risk than necessary to produce any given rate of return (Black and Litterman, 1992; Markowitz, 1952; Merton, 1972; Bodie and Marcus, 2005; Sharpe, 1964).

Most of the classic finance textbooks explais the efficient frontier from the perspective of the theoretical stand point but miss to address its practical implication. Students thus face the difficulties to fully grasp the theory due to the lack of application in the construction of an actual portfolio. To make the theory more understandable, often, we assign students a project asking students to create an efficient frontier in Excel. The general procedures required to create efficient frontiers rely on the application of the solver in Excel, beyond knowing the mathematical equations. However, when using this methodology in managing their own funds, students realize that the suggested investment strategy might not be as practical as expected since it sometimes suggests students to invest heavily in a couple of stocks only, leaving the rest unaffected. In this case study, we provide a modification of the conventional approach to deal with this problem. Meanwhile, we present a set of commands based on Matrix in Excel, not solver, to build efficient frontier. The subsequent template in Excel is available for interested readers.

The Case

Sharpe (1964) and Merton (1972) build a theoretical foundation for the efficient frontier. Several studies have tried to teach how to create an efficient frontier using Excel and Matlab. Girard and Ferreira (2005) and Caples, Michael, and Grady (2013) explain in detail how to use solver to create the efficient frontier. Chen et. al. (2010) and Roychoudhury (2007) provide a thorough review from the perspective of the theories and offer resolution of the efficient frontier in Matlab. In this study, we first provide a complete theoretical review of the efficient frontier. After that, we apply the theory in a real case setting.

In this case study, we create the efficient frontier based on the following eight stocks: Wal-Mart (WMT), Home Depot (HD), Citigroup (C), Southwest Airlines (LUV), Texas Instruments (TXN), Johnson and Johnson (JNJ), IBM (IBM), and Boeing (BA). All of the eight companies are not only big, but also acting as industry leaders. Next, we find the optimal combinations of those stocks for any investors with certain risk tolerance.

First, we collect the monthly stock price for each stock at Yahoo.com and calculate the annualized monthly stock returns and standard deviation by the following equations.

$$R_i = (1 + r_i)^{12} - 1 \quad (1)$$

$$\sigma_i = \sigma_{monthly_i} * \sqrt{12} \quad (2)$$

where r_i and R_i are the monthly and annual stock returns of stock i, respectively. σ_i is the annualized standard deviation.

After obtaining the annualized stock returns and risk, the portfolio's return and risk are calculated based on the following equations.

$$E(R_{port}) = \sum_i w_i * R_i \quad (3)$$

$$\sigma_{port} = \sqrt{\sum_j \sum_i w_i w_j \gamma_{i,j} \sigma_i \sigma_j} \quad (4)$$

Where γ_{ij} is the correlation between stocks i and j calculated as follows.

$$\gamma_{ij} = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{[n\sum x^2 - (\sum x)^2][n\sum y^2 - (\sum y)^2]}} \quad (5)$$

where $\gamma_{ij} = 1$ for i=j.

To construct the efficient frontier, we further set up the following rules.

Minimize σ_{port}

Subject to: $0 <= w_i$ for all security i

$w_i <= 1$ for all security i

$\Sigma w_i = 1$

$E(R_{port}) = X$

where w_i is the weight of stock i.

Using solver in Excel, we can find the optimal portfolios and draw the efficient frontier.
To find the weight, we can also use the following Lagrangian objective function.

$$C = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{Cov}(r_i r_j) + \lambda_1 \left[E^* - \sum_{i=1}^{n} w_i E(r_i) \right] + \lambda_2 \left(1 - \sum_{i=1}^{n} w_i \right) \quad (6)$$

Taking the partial derivatives with respect to each of the variables in the equation and setting the new equations equal to zero, we get the minimization of risk under the Lagrangian constraints (Chen, et. al, 2010; Roychoudhury, 2007). Working out those equations, we can find the weights of the optimal portfolios.

$$w^{T*} = \lambda \mu^T \Sigma^{-1} + k 1^T \Sigma^{-1} \quad (7)$$

$$\lambda = (B\mu_p - C)/D \quad (8)$$

$$k = (A - C\mu_p)/D \quad (9)$$

The solution of the optimal portfolio is the following.

$$\sigma_p^2 = (B\mu_p^2 - 2C\mu_p + A)/ D \quad (10)$$

where μ_p is the given portfolio's return.

$$A = \mu^T \Sigma^{-1} \mu,\ B = 1^T \Sigma^{-1} 1,\ C = 1^T \Sigma^{-1} \mu,\ D = AB - C^2 \quad (11)$$

In excel Matrix,
A = MMULT(MMULT(TRANSPOSE(return vector), MINVERSE(covariance matrix)), return vector))
B = MMULT(MMULT(TRANSPOSE(identity vector), MINVERSE(covariance matrix)), identity vector))
C =MMULT(MMULT(TRANSPOSE(identity vector), MINVERSE(covariance matrix)), return vector))

where ∑ represents the NxN variance covariance matrix of the n stocks; μ is a 1xN column vector of the expected returns; w represents a 1xN column vector of the portfolio weight; i represents a 1xN column vector of 1's. The Excel solution without solver is available for readers who are interested in this template.

After creating the efficient frontier, we look for the optimal risky portfolio. The straight line connecting the risk free rate and this optimal risky portfolio is called the Capital Asset Line (CAL hereafter). The modern portfolio theory explains that investors should invest in only the risk free rate and this optimal risky portfolio. Based on their risk tolerance, investors need to balance between the two securities (Black and Litterman, 1992; Markowitz, 1952).

To achieve this CAL line, we can draw a tangent line from the risk free rate in the y-axis to the efficient frontier. However, it is difficult to find this tangent line in that there is not a general equation for the efficient frontier, let alone finding its first derivative. From a different perspective, this tangent line connects the risk free rate and the optimal risky portfolio, therefore, the optimal portfolio should have the best reward to risk - i.e., the highest Sharpe ratio.

$$\text{Max Sharpe Ratio} = \text{Max}((r_p - r_{rf}) / \sigma_p) \quad (12)$$

where r_p and σ_p are the optimal risky portfolio's return and standard deviation, respectively.
Since the Sharpe ratio keeps increasing, and then starts to taper off and finally decreases, we can locate this optimal risky portfolio on the efficient frontier by setting the equation of Sharpe Ratio in solver and then finding its highest value.

Assume that the market portfolio's return and standard deviation are known. Thus, investors should be holding the risk free rate and the optimal risky portfolio.

$$r_i = w_{rf} * r_{rf} + w_p * r_p \quad (13)$$

$$\sigma_i^2 = w_{rf}^2 * \sigma_{rf}^2 + w_p^2 * \sigma_p^2 \quad (14)$$

where r_i and σ_i are the return and standard deviation of investor i.

The CAL line means the following.

$$r_i = r_{rf} + ((r_p - r_{rf}) / \sigma_p) * \sigma_i \quad (15)$$

The last step is to find each investor's optimal way to invest given her risk tolerance degree. In other words, we need to figure out the weight of risk free rate and that of the optimal risky portfolio in her portfolio. To do so, we use the utility function that is widely applied in the literature.

$$U = E(r_i) - 0.005\, A\, \sigma^2_i \quad (16)$$

where U is the investor's utility and A is associated with an investor's risk aversion degree.

The tangent point of the indifference curve on CAL should be the optimal portfolio for an investor with a risk aversion level of A.

The optimal portfolio means that it has the highest utility, so

$$\text{Maximize } U = r_i - 0.005A\,\sigma^2_i, \text{ subject to } w_{rf} + w_p = 1 \qquad (17)$$

where r_{rf} is the risk free rate, w_{rf} and w_p are the weights of risk free rate and the optimal risky portfolio, respectively. Plugging (10) and (11) into (14), utility function can be expressed as follows.

$$U = w_{rf} * r_{rf} + (1-w_{rf}) * r_p - 0.005A\,\sigma^2_p * (1-w_{rf})^2 \qquad (18)$$

Forcing the first derivative of the above utility function to zero, we can calculate the optimal weights for any given A.

$$w_{rf^*} = (r_{rf} - r_p + 0.01A\,\sigma^2_p)/(0.01A\,\sigma^2_p) \qquad (20)$$

$$w_p^* = 1 - w_{rf} \qquad (21)$$

Case Results

The general steps of the Markowitz portfolio problem involve the following three steps. (1) Create efficient frontier. (2) Plot the Capital Asset Line (CAL), the tangent line to the efficient frontier from the risk free security on the y-axis. (3) Find the tangent point of the indifference curve on the CAL. This point tells the optimal way to invest.

In this exercise, we create an efficient frontier based on eight securities from different industries: Wal-Mart (WMT), Home Depot (HD), Citigroup (C), Southwest Airlines (LUV), Texas Instruments (TXN), Johnson and Johnson (JNJ), IBM (IBM), and Boeing (BA).

The following figure 1 illustrates the efficient frontier, the CAL, and the indifference curve.

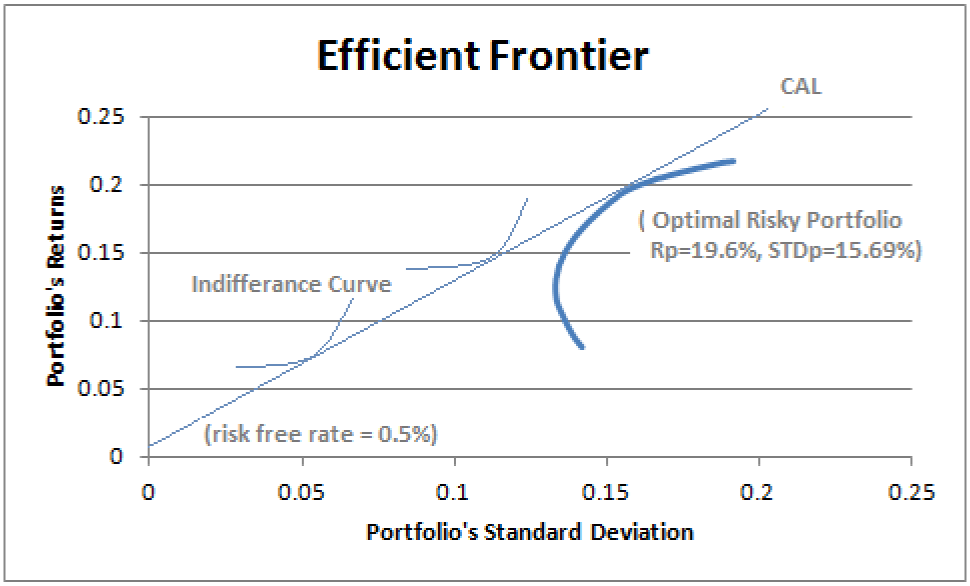

Figure 1: Efficient frontier, CAL, and Indifference Curve

Based on equations (6), (7), (8), and (9), we find the optimal risky portfolio. Its return and standard deviation are 19.6% and 15.69%, respectively. The portfolio is constituted by 47.73% in WMT, 20.95% in HD, and 31.32% in IBM. The results, based on solver are similar.

The CAL can be written as follows.

$$r_p = 0.5\% + ((19.6\% - 0.05\%) / 15.69\%) * \sigma_p \qquad (22)$$

Next, based on equations (16) and (17), we can find the optimal investment strategy for each investor given her risk aversion.

$$w_{rf^*} = (r_{rf} - r_m)/(0.01A\ \sigma^2_m) + 1 = -19.1/(0.01A*15.96^2) + 1 \qquad (23)$$

$$w_{m^*} = 1 - w_{rf} = 19.1/(0.01A*15.96^2) \qquad (24)$$

where A indicates an investor's risk aversion. The higher the index A is, the more aversion the investor is, and therefore, she should hold more risk free assets and less risky portfolio.

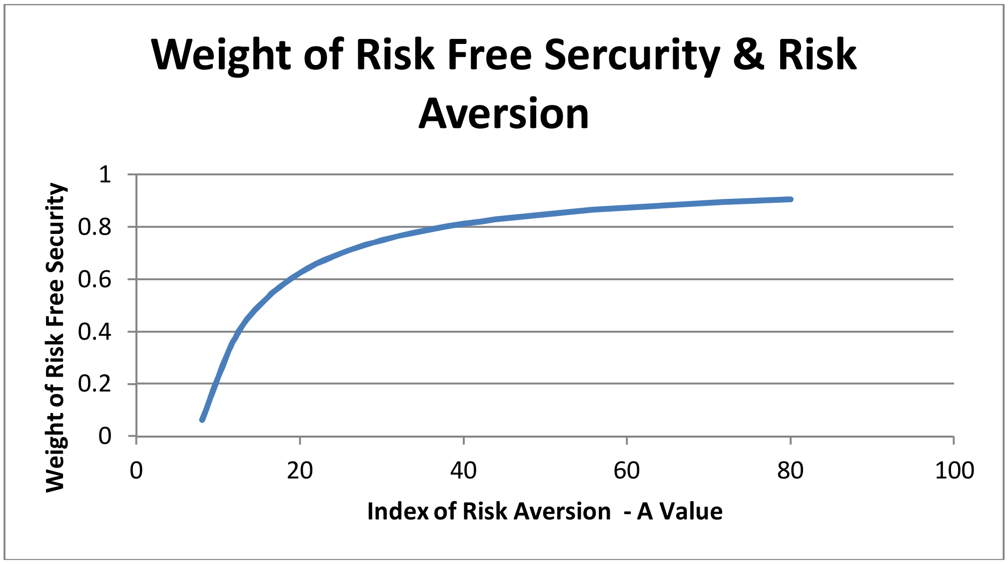

Figure 2: Risk aversion and Fund Allocation

Figure 2 shows that investors with different tolerance of risk will hold different portfolios. For instance, when A equals to 20, investors should invest 62.5% of their total funds in the risk free asset and the rest 37.5% in the optimal risky portfolio. In comparison, when the A value is 12, investors should just do the opposite, which is 37.5% in the risk free security and 62.5% of the optimal risky portfolio. The template based on Matrix is available to the interested reader.

Discussion of the heavy concentration stocks

It is likely that the solutions of the optimal portfolios have heavy concentration stocks, especially when international stocks are included and when the number of stocks is large. In the next example, we pick a total of 38 stocks from the blue chip sector, the financial sector, and the technology sector. Among the 38 stocks, 3 are from China. The stock information is in

Refer Table I (Exhibits)

The efficient frontier based on the above stocks is shown in figure 3.

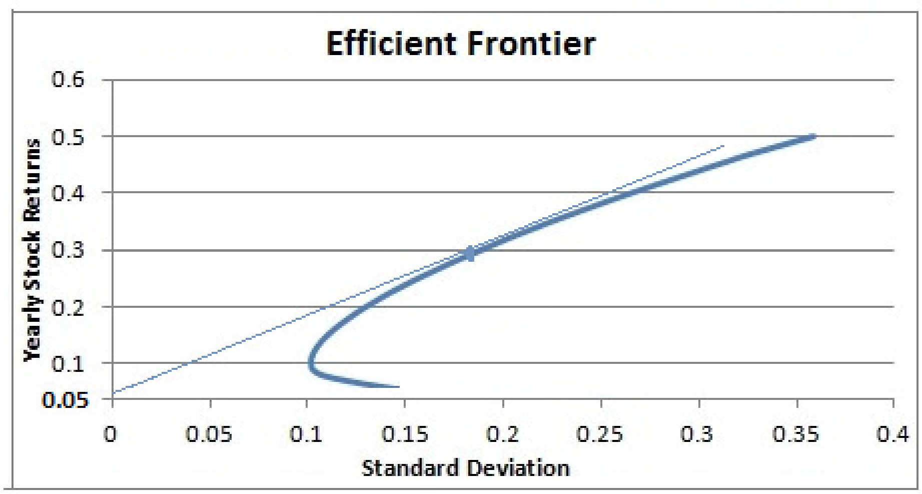

Figure 3: Efficient Frontier based on 35 stocks of U.S. and 3 stocks in China

Table II shows the weights of the optimal portfolios on the efficient frontier. Figure 4 visualizes the results in Table II.

Refer Table II (Exhibits)

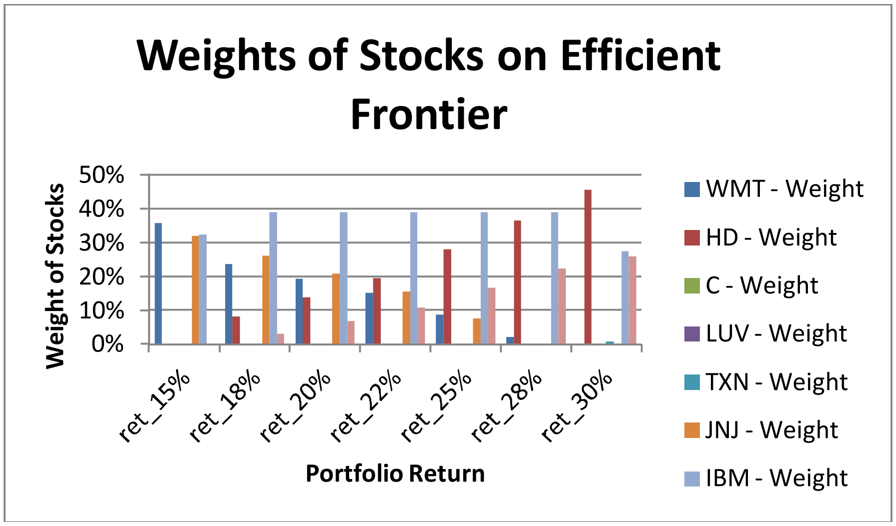

Figure 4: Weights of stocks on efficient frontier

As shown, the results are dominated by several key stocks only and many stocks' weights are very low or even zero. For instance, one of the weights suggests that we put 42% of our portfolio into Apple alone. We may not be comfortable with such a heavy allocation, and we thereby impose the additional constraint that no single asset in our portfolio takes up more than 15% of the total fund.

The comparison between the efficient frontier with and without the newly added constraints is shown in the following figure 5.

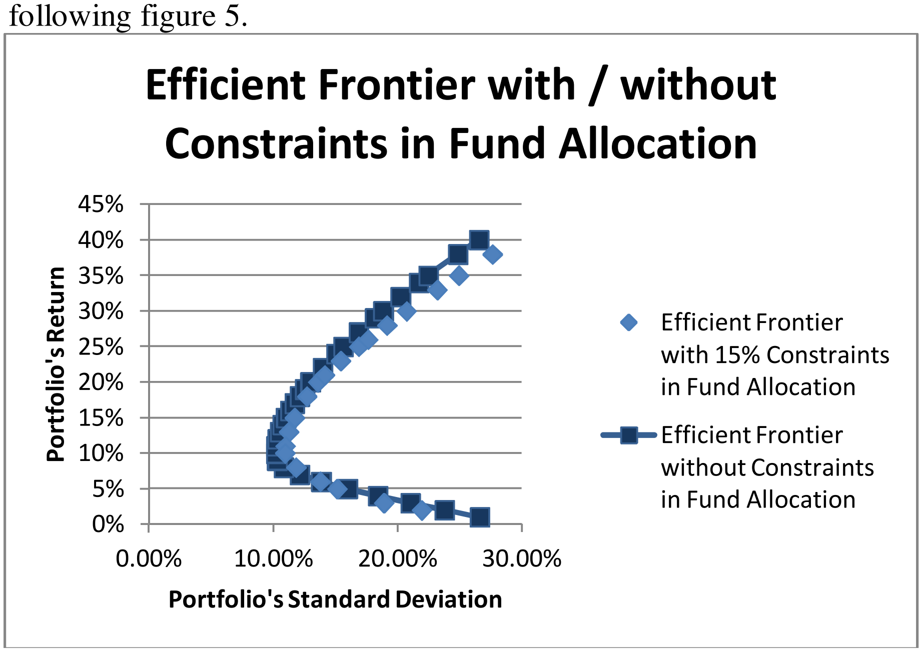

Figure 5: Efficient frontier with / without constraints in fund allocation

By adding the 15% limit on the fund allocation, we put constraints on the optimization process, causing portfolios to be relatively less efficient than the unconstrained optimization and thereby causing the efficient frontier to shrink. However, this should not be of concern as long as the constraints reflect the needs of the investor.

Conclusion

In this study, we review the efficient frontier theoretically. We also provide a template in Excel based on Matrix for the efficient frontier. Further, we offer a solution for students when dealing with heavy concentrated stocks. Our students use this case study as studying materials for efficient frontier. So far, the feedbacks are positive.

Reference

Bodie, Z., A. Kane, and A. J. Marcus, 2010, Investment (McGraw-Hill, Columbus, OH.)

Caples, Stephen C., Hanna, Michael E., and Perdue, Grady, 2013, Ascertaining the efficient frontier in the classroom, *Journal of Economics and Economic Education Research* 14(1), 97.

Chen, Weipeng, Huimin Chung, Kengyu Ho, and Tsuiling Hus, 2010, Portfolio optimization models and mean-variance spanning tests, Handbook of Quantitative Finance and Risk Management pp 165-184.

Girard, Eric, and Eurico Ferreira, 2005, A n-assets efficient frontier guideline for investments courses, *Journal of College Teaching & Learning* 2(1), 53-65.

Markowitz, Harry, 1952, Portfolio selection, *Journal of Finance* 7(1), 77-91.

Merton, Robert, 1972, An analytical derivation of efficient frontier, *The Journal of Financial and Quantitative Analysis* 7(4), 1851-1872.

Saurav Roychoudhury, 2007, The optimal portfolio and the efficient frontier, NSF funding paper series.

Sharpe, William, 1964, Capital asset prices: A theory of market equilibrium under conditions of risk, *Journal of Finance* 19(3), 425-442.

Authors

Maggie Foley*

Assistant Professor of Finance, Davis College of Business, Jacksonville University, Jacksonville, FL, 32256, mfoley3@ju.edu

Barry Thornton

Professor of Economics, Davis College of Business, Jacksonville University, Jacksonville, FL, 32256, bthornt@ju.edu

Biqing Huang

Assistant Professor of Finance, College of Business, Angelo State University, San Angelo, TX, 76909, biqing.huang@angelo.edu

Xiaowei Liu

Associate Professor of Finance, College of Business, St. Ambrose University, Avenport, IA, liuxiaowei@sau.edu

*Corresponding Author

Exhibits

Table I: Monthly Return, Standard Deviation, and Return per Unit of Risk

Categories	Companies	Monthly Return	Standard Deviation	Return / Risk
U.S. Stocks	Alcoa Inc.	-0.002	0.111	-0.014
	Apple Inc.	0.033	0.103	0.317
	Microsoft	0.007	0.07	0.104
	American Express Co.	0.012	0.115	0.108
	The Boeing Co.	0.014	0.076	0.182
	United Technologies Corp.	0.011	0.056	0.192
	Bank of America	0.003	0.15	0.022
	Citi Group Inc.	-0.006	0.156	-0.037
	Goldman Sachs	0.01	0.095	0.109
	JP Morgan Chase & Co.	0.01	0.089	0.112
	Caterpillar Inc.	0.013	0.101	0.128
	Chevron Corp.	0.012	0.058	0.215
	Exxon Mobil Corp.	0.01	0.054	0.179
	E. I. DU PONT DE NEMOURS & CO.	0.009	0.077	0.112
	Disney	0.013	0.064	0.196
	General Electronic Co.	0.004	0.085	0.049
	3M Company	0.009	0.058	0.151
	General Motors Co.	0.009	0.095	0.096
	Google	0.021	0.095	0.218
	IBM	0.009	0.055	0.161
	Netease Inc.	0.024	0.117	0.201
	The Home Depot Inc.	0.01	0.066	0.157
	Target	0.006	0.07	0.079
	Wal-Mart	0.006	0.046	0.14
	Hewlett Packard Co.	0.008	0.088	0.087
	Intel Corp.	0.006	0.073	0.082
	Johnson & Johnson	0.007	0.04	0.167
	Merck & Co. Inc.	0.01	0.068	0.147
	Pfizer Inc.	0.006	0.059	0.11
	The Coca Cola Company	0.01	0.044	0.22
	McDonalds	0.014	0.045	0.305
	Mondelez International Inc.	0.008	0.052	0.149
	The Proctor & Gamble Co.	0.007	0.044	0.155
	AT&T Inc.	0.009	0.051	0.169
	Verizon	0.008	0.053	0.158
Chinese Stocks	Baidu	0.043	0.156	0.275
	Sina Corp.	0.018	0.134	0.132
	Sohu.com Inc.	0.025	0.157	0.159

Table II: Allocation of Funds of the Optimal Portfolios

	Inefficient Portfolios		Efficient Portfolios								
Return	0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
Standard Deviation	0.2661	0.1597	0.1019	0.1100	0.1296	0.1562	0.1883	0.2246	0.2653	0.3093	0.3587
WalMart	*41.48%*	*37.22%*	*26.64%*	*22.20%*	*18.19%*	*10.43%*	*0.00%*	*0.00%*	*0.00%*	*0.00%*	*0.00%*
Apple	*0.00%*	*0.00%*	*0.00%*	*5.23%*	*10.79%*	*15.21%*	*20.92%*	*27.47%*	*34.99%*	*42.47%*	*40.22%*
McDonald	*0.00%*	*0.00%*	*8.18%*	*24.15%*	*34.36%*	*41.12%*	*41.33%*	*37.15%*	*21.71%*	*5.98%*	*0.00%*
Baidu	*0.00%*	*0.00%*	*0.00%*	*0.00%*	*2.77%*	*7.38%*	*12.02%*	*17.20%*	*23.52%*	*29.67%*	*44.51%*
Johnson	11.81%	24.71%	11.63%	5.09%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Merck	0.00%	0.00%	0.82%	2.11%	3.22%	2.68%	2.26%	0.00%	0.00%	0.00%	0.00%
Coca Cola	0.00%	0.00%	1.57%	3.40%	3.97%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Mondeke	0.00%	0.00%	4.32%	5.02%	3.70%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
P&G	4.04%	18.05%	15.53%	6.30%	1.17%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
ATT	0.00%	0.00%	7.51%	3.74%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Verison	0.00%	0.00%	1.49%	1.68%	4.34%	3.46%	1.83%	0.00%	0.00%	0.00%	0.00%
IBM	0.00%	0.00%	8.58%	2.20%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Home Depot	0.00%	0.00%	0.84%	4.58%	6.30%	6.69%	6.66%	0.00%	0.00%	0.00%	0.00%
Google	0.00%	0.00%	0.00%	1.82%	1.71%	1.02%	0.55%	0.00%	0.00%	0.00%	0.00%
Netease	0.00%	0.00%	1.01%	6.55%	9.36%	12.01%	14.43%	18.17%	19.77%	21.88%	15.27%
Exxon	0.00%	0.00%	11.63%	5.94%	0.13%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Alcoa	0.02%	0.00%	0.00%	0.00%	0.01%	0.02%	0.02%	0.01%	0.01%	0.01%	0.00%
Target	0.00%	2.86%	0.25%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Citi	42.65%	17.16%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%