



Research Article

Construction and Qualitative Analysis of Mathematical Model for Biological Control on Cereal Aphid Population Dynamics

Judith J. E. J. Ogal¹, N. B. Okelo^{1*}, Roy Kiogora², Thomas Onyango³

¹School of Mathematics and Actuarial Science,
Jaramogi Oginga Odinga University of Science and Technology,
P. O. Box 210-40601, Bondo-Kenya.

²Department of Pure and Applied Mathematics,
Jomo Kenyatta University of Agriculture and Technology, Kenya.

³Department of Industrial and Engineering Mathematics,
The Technical University of Kenya, Kenya.

*Corresponding author's e-mail: bnyaare@yahoo.com

Abstract

Cereal farming is a major economic activity for farmers in most parts of the world. In Kenya, where the agricultural sector is the backbone of the economy, cereal production is a major source of income to the farmers as it is used for both human and livestock consumption. A common sight in cereal crop farms is cereal aphids whose population has been on the rise, aided by various environmental factors that may have favoured their increase. The frequent outbreak of aphids and the extent of damage they cause on these farms have laid precedence to undertake studies aimed at understanding their population dynamics. This study analysed the impact of biological control on cereal aphid population. The study developed a mathematical model of the impact of predation on cereal aphid's population which can project stable systems of control. It also determined the extent of effectiveness of the model by comparing after modification, stability of the models. Two sets of models based Rosenzweig-MacArthur prey-predator were developed, through adjusting the function representing the prey-predator interaction. It was determined that one model demonstrated the ability to capture a more accurate analysis of data compared to the other. After finding the local stability of each, a suitable Lyapunov function was developed and used to analyze the global stability of the system.

Keywords: Qualitative analysis; Mathematical model; Biological control; Cereal aphid; Population dynamics.

Introduction

Cereals include wheat, rice, maize, barley, oats, rye, millet, sorghum, buckwheat and mixed grains [1-4]. Cereals are important crops in the world and the apparent increase of cereal pests and diseases over the past years has necessitated the need to study cereal pests. Any loss of yields caused by pests has serious consequences, both locally and internationally [5-9]. Cereal aphids can cause considerable losses to yield in some years. Their abundance usually varies from year to year [3] and from place to place [10]. For effective advisory service knowledge of loss of yields relative to aphid density and the growth of aphid population, are needed [3] Cereal crops are seriously affected by different species of aphids. According to [11], the most common ones are *Rhopalosiphum padi* (Oat Bird-Cherry, Bird

Cherry Oat Aphid), *R. maidis* (Corn Leaf Aphid), *Sitobion avenae* / *Syn Macrosiphu avenae* (Grain Aphid / English Grain Aphid), *Schizaphis graminum* (Green bug), *Metopolophium dirhodum* / *Syn Acrythosiphon dirhodum* (Rose Wheat Aphid / Rose Grass Aphid) and *Diuraphis noxia* (Russian Wheat Aphid).

There have been several studies undertaken to understand population dynamics of cereal aphids, by looking at their lifecycles. Others have taken keen interest on the impact pest control has on them. According to [11] the damage cereal aphid does to crops include: a) transmit virus diseases which cause weakened plants and reduced yields; b) feed on shoots of host plants; c) suck the sap from leaves which then curl and wilt; d) feed on leaves of crops causing yellowing and other phytotoxic effects;

e) feed on developing ears of crops; f) some produce a toxin which may kill young plants. This damage on crops has brought about the need to control cereal aphid's population.

The methods used for controlling cereal aphids are mainly divided into three categories: chemical control, which involves use of insecticides; biological control for example breeding resistant crops, release of predators [12] and cultural control which involves management of the physical or biological environment of the crop, an example is crop rotation. The method chosen for pest control and management has to be considered carefully.

Chemical insecticides are the most common method of pest control in agriculture. On one hand, the disadvantages of pesticides are [13] a) increased resistance by pest resulting into reduction of efficiency; b) high negative impact over beneficial insect population; c) new and more pest surges; d) reduction of natural pest control due to destruction of natural enemies; e) incidence of secondary pests; f) chemical residues in crops; g) long term chemical residues in the agricultural ecosystem; h) ecological accidents; i) long term ecological residues in the agricultural ecosystem; j) high number of accidents that intoxicate human beings and in some cases result into death. On the other hand, in the presence of optimal conditions under which aphids have the ability to rapidly multiply, the use of insecticides may be necessary. Biological control is often effective when aphid number is low [14-17].

The specific objectives of this study have been: To develop a mathematical model(s) that can show the effect of biocontrol on aphid population dynamics, and to carry out qualitative analysis of the models.

Research methodology

In this study, we have extended the mathematical background given by Rosenzweig-MacArthur prey-predator model using the work done by [7]. We have formulated two sets of Rosenzweig-MacArthur prey-predator model with one predator and the prey, and then solve them analytically. The second set of the model seeks to modify and thus give a more accurate analysis of data compared to the first set of the model

Results and discussions

Cereal aphid population dynamics

We adopt the model developed by [7-9,14] that shows in a simplified form the dynamics of aphid population dynamics. This model is a combination of a logistic model with a variable 'carrying capacity' and a cumulative density model. It is given by:

$$\frac{dn}{dt} = ax, b(0) = 0, \quad (1)$$

$$\frac{dx}{dt} = (r - b)x \left(1 - \frac{x}{k}\right), \quad x(0) = x_0, \quad (2)$$

The carrying capacity is assumed to be varying between k_{max} and k_{min} following a cosine function,

$$k(t) = \{k_{max} - k_{min}\} \times \{(\cos(\pi t) + 1) / 2\} + k_{min} \quad (3)$$

Where k is the carrying capacity, $b(t)$ is the cumulative density of aphids at time t , $x(t)$ is the density of aphids at time t , a is the scaling constant relating aphid cumulative density to its own dynamics, and r the maximum potential growth rate of the aphids. In the logistic model with variable carrying capacity that has been incorporated above, the distinctive feature of the aphid dynamics is that the decline in numbers mainly arises by its own dynamics rather than by other species.

The varying nature of the soluble nitrogen in the host plants, which is a good indicator of host quality, explains the assumption of the varying carrying capacity. On the other hand, in the cumulative density model incorporated above, the assumption is that the cumulative density is the regulatory term that slows down the instantaneous rate of increase. This is pegged on the assumption that it is the sum of the number of individuals multiplied by their life span, which determines the slowing down of the instantaneous rate of increase. Thus it could influence food quality and hence slow down population rate of increase. The advantage of the Kindlmann model is that it gives the most flexible model. However, its limitation is that there is the problem of how to measure the time varying carrying capacity.

Biological control

The focus of this project is predation as a means of pest control. We particularly look at Coccinellidae (ladybird beetles) as our predator.

Coccinellid (ladybird beetles)

Many studies have been conducted on ladybirds as aphid predators because they are visible and also have economic importance to a variety of crops. The larvae and adults of coccinellids feed on the same type of prey species and occur in identical habitats [18]. The lifetime fecundity of Coccinellid varies greatly between species and may range from slightly more than 100 to more than 1500 eggs per female. Developmental times varies greatly between species and is influenced by temperature, the amount of food consumed and prey species. Coccinellids usually search their environment for plants with aphid prey randomly rather than systematically, as demonstrated by their frequent returns to the same site [4]. When searching for oviposition sites, females respond to the amount and quality of the prey they encounter. In this way, adults ensure that aphids will not become scarce before the larvae complete their development. Ladybirds seem to avoid laying eggs in colonies with high risk short term extinction. However, a certain threshold of aphid density seems to be necessary to elicit adult oviposition behaviour [19].

The presence of potential competitors may also negatively affect oviposition decision. Most aphidophagous ladybird species are able to develop on a variety of aphid prey. There are, however, size constraints. Small species consume small aphids and early instar ladybird larvae prefer early larval stages of their prey species. There is controversy on how large the impact of ladybird is, in reducing aphid populations. On one hand, many species of coccinellid share several characteristics of successful predators, such as high searching capacity, high voracity, appropriate food range, and the capacity to develop on alternative food if aphids are scarce [6] implying that ladybirds are able to rapidly reduce high aphid densities, and especially if both aphid and ladybird peaks coincide. On the other hand, it has been argued that a lack of synchronization and the restriction to one or to two generations per year, limits the efficiency of ladybirds in biological control [7]. Nevertheless, coccinellids are important predators of aphids, especially in cereals and in maize, where they contribute to a significant reduction in populations of economically important aphids [20].

Prey-Predator model description

To investigate the effect of ladybird predation on cereal aphid's population, we develop two sets of models. The two sets of models can be called Case I and Case II for the first and the second set of models respectively. The difference between the two sets of models arises from the functional response term, representing the interaction between the predator and the prey. In Case I, the interaction is given by $\frac{aNP}{1+ahN}$, whereas in Case II it is given by $\frac{aNP}{1+ahN+N}$. This implies that the variable N , which is the prey density, has been introduced, and added to the denominator value in Case II. Case II Model actually modifies and improves Case I Model. The two sets of models are then used to analyse the effect of the predator on the prey, and a comparison is then made to see how effective Case II Model is, more than the Case I Model. So, we expect Case II model to give a more accurate analysis of data as compared to Case I model. We use the Rosenzweig-MacArthur predator-prey model which has a predator depending on the prey. The model we develop shows the logistic growth and the cumulative density, b , of the prey.

The predator's increment rate depends on its growth rate μ , and its conversion efficiency (e), as exhibited in the function that represents prey-predator interaction. All these characteristics of the predator and prey are incorporated in this model to study their dynamics. This model has a Holling Type II functional response, that is, the number of prey consumed per predator initially rises as the density of the prey increases but then levels off with further increase in predator population.

Prey-Predator model characteristics

The prey-predator model usually exhibits the following characteristics: the prey and the predator's populations are periodic in nature; an increase in the prey's population is followed by an increase in the predator's population or a decrease in the prey population is followed by a decrease in the predator population. When we vary (either as small or large) the population of variables, N and P , in the functional response term in either of the two Cases, $\frac{aNP}{D+N}$ or $\frac{aNP}{D+N+P}$ we observe a number of characteristics that

apply to both Models. The variables N and P can be varied, these variations in turn have an effect of either increasing or decreasing the population of either the prey or the predator. The variations can take the form:

- i.) N -small and P -large
- ii.) N -large and P -large,
- iii.) N -small and P -small and
- iv.) N -large and P -small.

Hence, a summary of characteristics that arise from the variation of these two parameters as pointed out above include:

- a.) A small number in predators results in an increase in prey numbers;
- b.) A large number in predators results in a decrease in prey numbers;
- c.) A large number of prey results in increases in predators, and
- d.) A small number of preys results in a decrease in predators.

The periodic nature of the predator-prey populations often results into a situation whereby the peak of prey population occurs some time before the predator's population peak.

Assumptions of the model

The following assumptions are made to formulate the model:

- i. There is a logistic growth of the prey in the absence of the predator with carrying capacity, K and the intrinsic growth rate, r .
- ii. The predator consumes the prey according to the functional response $\frac{aNP}{D+N}$ for Case I Model and $\frac{aNP}{D+N+P}$ for Case II Model and grows logistically with intrinsic growth rate μ .
- iii. Predator would go extinct if prey were absent, due to lack of prey.
- iv. The proportional rate of increase of prey decreases as the number of predator increase.
- v. The growth rate of the predator increases with the increase in the number of prey.

The Prey-Predator model formulation

The model formed from the interaction between the prey and predator is as follows [8].

Case I:

$$\left. \begin{aligned} \frac{dN}{dt} &= N(r - b) \left(1 - \frac{N}{K}\right) - \frac{aNP}{D + ahN} \\ \frac{dP}{dt} &= \mu P - e \frac{aNP}{D + ahN} \end{aligned} \right\} (4)$$

$$\frac{dP}{dt} = \mu P - e \frac{aNP}{D + ahN}$$

Case II:

$$\left. \begin{aligned} \frac{dN}{dt} &= N(r - b) \left(1 - \frac{N}{K}\right) - \frac{aNP}{D + ahN + P} \\ \frac{dP}{dt} &= \mu P - e \frac{aNP}{D + ahN + P} \end{aligned} \right\} (5)$$

Without loss of generality we simplify the models by taking, $ah = 1$. Hence the above equations (4) and (5) respectively in Case I and Case II respectively can be written as indicated in equations (6) and (7) below,

Case I:

$$\left. \begin{aligned} \frac{dN}{dt} &= N(r - b) \left(1 - \frac{N}{K}\right) - \frac{aNP}{D + N} \\ \frac{dP}{dt} &= \mu P - e \frac{aNP}{D + N} \end{aligned} \right\} (6)$$

Case II:

$$\left. \begin{aligned} \frac{dN}{dt} &= N(r - b) \left(1 - \frac{N}{K}\right) - \frac{aNP}{D + N + P} \\ \frac{dP}{dt} &= \mu P - e \frac{aNP}{D + N + P} \end{aligned} \right\} (7)$$

Where $N > 0$ and $P > 0$, respectively. This implies that all the parameters in the model are positive. We then perform non-dimensionalization to reduce the number of parameters in the model in equation (6) and (7) by reducing \bar{t} , \bar{N} and \bar{P} into non-dimensional form using,

$$\bar{t} = \frac{t}{r}, \bar{N} = \frac{N}{K}, \bar{P} = \frac{P}{K}$$

Then, further by setting the parameters $\bar{a} = \frac{aK}{r}$, $\bar{\mu} = \frac{\mu}{r}$, $\bar{K} = \frac{K}{r}$ then dropping the sign, we find that the equations (4) and (5) take the form in equations (6) and (7) respectively.

Case I

$$\frac{d\bar{N}}{d\bar{t}} = \bar{N} \left(1 - \frac{\bar{N}}{\bar{K}}\right) (1 - \bar{N}) - \frac{\bar{a}\bar{N}\bar{P}}{\bar{D} + \bar{N}} \quad (8a)$$

$$\frac{d\bar{P}}{d\bar{t}} = \bar{\mu}\bar{P} - e \frac{\bar{a}\bar{N}\bar{P}}{\bar{D} + \bar{N}} \quad (8b)$$

Case II

$$\frac{d\bar{N}}{d\bar{t}} = \bar{N} \left(1 - \frac{\bar{N}}{\bar{K}}\right) (1 - \bar{N}) - \frac{\bar{a}\bar{N}\bar{P}}{\bar{D} + \bar{N} + \bar{P}} \quad (9a)$$

$$\frac{d\bar{P}}{d\bar{t}} = \bar{\mu}\bar{P} - e \frac{\bar{a}\bar{N}\bar{P}}{\bar{D} + \bar{N} + \bar{P}} \quad (9b)$$

$N(0) > 0$ and $P(0) > 0$, respectively.

The effect of varying parameters N and P on prey/ predator population

The parameters in our analysis of the prey-predator interaction, which we have gotten from

a previous study, give us fixed values for these parameters. However, the time span and the densities of the prey/ predator vary. The variation of initial prey's density, N , or the initial predator's density, P , have an effect on the both the prey and predators population. We explore four possible variations illustrated as (i), (ii), (iii) and (iv), below:

i. N-small, P-large

When the number of predators largely exceeds that of prey, the result is a decrease in prey population and a decrease in predator population. This is because more predators will be depending on a small number of prey for food, implying that the predator's consumption rate is higher than the prey's growth (increase) rate.

ii. N-large, P-small

Given that, the number of both preys and predators are both large. This results into an increase in the population of both the prey and the predator. The number of predator is directly proportional to the number of prey, so when the prey increases the predators also increases because of the dependence.

iii. N-small, P-small

When the number of the number of both preys and predators is small, there will be a decrease in the population of both the prey and the predator. The number of predator is directly proportional to the number of prey, so when the prey decreases the predators also decreases because of the dependence.

iv. N-large, P-small

When the prey largely outnumbers the predators, the result is an increase in prey population and an increase in predator population. This is because the number of prey on which the predators feed is already high, the growth rate of predators increase because there is enough prey to feed on.

These variations on the prey populations and predator numbers in the prey-predator model have an oscillatory character.

Qualitative analysis of the model

Equilibrium points of the model

We determine the conditions for the existence of equilibrium points of the two systems of equations. The two sets of equations in Case I and that in Case II are almost similar, therefore the analysis done below will apply to

both. The equilibrium point of $E_0(0, 0)$ is trivial, therefore we will not dwell on it. When we equate equations (8) and (9) to zero, we find that the system has three equilibria, that is; $E_1(1, 0)$, $E_2(0, 1)$ and $E_3(1, 1)$.

i. Existence of $E_1(\bar{N}, 0)$ with $\bar{N} > 0$.

Let $P = 0$. Equation (6) gives:

$$N \left(1 - \frac{b}{r} \right) (1 - N) = 0$$

From this we have $N=1$, which implies that $N = K$. Thus in the absence of predator P , the prey population N increases until it reaches the carrying capacity K .

ii. Existence of $E_2(0, \bar{P})$ with $\bar{P} > 0$.

Let $N=0$. Equation (6) gives:

$$\mu P = 0,$$

Since $P > 0$, it implies $\mu = 0$. This means that the growth rate of the predator is zero. If the growth rate is zero, there can be no increase in predator's population. Therefore the predator's population will tend to extinction when there is no prey.

iii. Existence of $E_3(\bar{N}, \bar{P})$ with $\bar{N} > 0$ and $\bar{P} > 0$.

Equation (6) gives:

$$N \left(1 - \frac{b}{r} \right) (1 - N) - \frac{aNP}{D + N} = 0$$

and

$$\mu P - \frac{aNP}{D + N} = 0$$

From these we have,

$$\bar{N} = \frac{\mu}{a} (D + N) \quad \text{and}$$

$$\bar{P} = \frac{1}{a} \left[(D + N) \left(1 - \frac{b}{r} \right) \left(1 - \frac{\mu}{a} (D + N) \right) \right]$$

Thus, $E_3(\bar{N}, \bar{P}) =$

$$E_3 \left(\frac{\mu}{a} (D + N), \frac{1}{a} \left[(D + N) \left(1 - \frac{b}{r} \right) \left(1 - \frac{\mu}{a} (D + N) \right) \right] \right)$$

This exists if, $b < r$ and $\mu < \frac{a}{D+N}$. Therefore the equilibrium exists if,

$$b < r \tag{10}$$

$$\mu < \frac{a}{D+N} \tag{11}$$

Condition (10) implies that prey's cumulative density b , which is the regulatory term, must be less than its growth rate r , whereas condition (11) implies that, $\mu < \frac{a}{D+N}$, that is, the predator's mortality rate must be less than the quotient of the maximum killing rate the sum of saturation-constant and the population of the prey.

Therefore, $E_3 (\bar{N}, \bar{P})$ exists only if conditions (10) and (11) are in place.

Local stability of the equilibrium points

The system of equations for our model is nonlinear therefore we use the Jacobian matrix which enables us to linearize the system and determine the characteristic equation. We get the characteristic equation by calculating $|J - \lambda I|$ where J is the Jacobian matrix and λI the identity matrix. We then find the roots of the characteristic function which enables us to determine the stability of the equilibrium solution.

The stability of each equilibrium point is studied by computing the Jacobian matrix and finding the eigen values evaluated at each equilibrium point. We only focus on finding the local stabilities for then on-trivial equilibrium points. If all real eigen values are negative then the equilibrium point is stable. If there is a positive eigenvalue or an eigenvalue with a positive real part, then the equilibrium is unstable. From equations (7), the Jacobian matrix is given by

$$J(E_i) = \begin{bmatrix} \frac{\partial f_1}{\partial N} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial N} & \frac{\partial f_2}{\partial P} \end{bmatrix} \tag{12}$$

When we work out the values of each individual element in the Jacobian matrix we get,

$$J(E_i) = \begin{bmatrix} A^{**} & -\frac{aN}{(D+N)} \\ -\frac{aP}{(D+N)^2} & \mu - \frac{aN}{(D+N)} \end{bmatrix} \tag{13}$$

Where the value of,

$$A^{**} = \left(1 - \frac{b}{r}\right) (1 - 2N) - \frac{aP}{(D+N)^2}$$

The local stability for each equilibrium point is analysed as follow:

- i. $E_1 (\bar{N}, 0) = E_I(1, 0)$. The Jacobian matrix evaluated at E_I gives

$$J(E_1) = \begin{bmatrix} \left(\frac{b}{r} - 1\right) & -\frac{aN}{(D+N)} \\ 0 & \mu - \frac{aN}{(D+N)} \end{bmatrix} \tag{14}$$

We then take the determinant of the matrix and subtract the identity matrix λI to obtain

$$|J(1, 0) - \lambda I| = \begin{vmatrix} \left(\frac{b}{r} - 1\right) - \lambda & -\frac{aN}{(D+N)} \\ 0 & \mu - \frac{aN}{(D+N)} - \lambda \end{vmatrix}$$

The eigen values of the matrix $J(E_I)$ are $\lambda_1 = \left(\frac{b}{r} - 1\right)$ and $\lambda_2 = \mu - \frac{aN}{(D+N)}$.

These eigen values are negative if, $b < r$ and $\mu(D+N) < a$. If eigen values λ_1 and λ_2 are both negative, the equilibrium is stable.

If $b < r$ and $a < \mu(D+N)$ then at least one of the two eigen values is positive. If at least one of the eigen values is positive, the equilibrium becomes unstable.

- ii. $E_2(0, \bar{P}) = E_2(0, 0)$

The Jacobian matrix evaluated at E_2 is

$$J(E_2) = \begin{bmatrix} \left(1 - \frac{b}{r}\right) & 0 \\ 0 & \mu \end{bmatrix} \tag{15}$$

Taking the determinant of the matrix and subtracting the identity matrix λI we obtain

$$|J(0, 0) - \lambda I| = \begin{vmatrix} \left(1 - \frac{b}{r}\right) - \lambda & 0 \\ 0 & \mu - \lambda \end{vmatrix}$$

The eigen values of the matrix $J(E_2)$ are $\lambda_1 = \left(1 - \frac{b}{r}\right)$ and $\lambda_2 = \mu$. Since, $\lambda_2 = \mu$, and from existence of equilibrium points, we found out that $\mu = 0$, implying that $\lambda_2 = 0$. We also know that $b < r$ which means that the eigenvalue λ_1 is positive, hence the equilibrium is unstable.

- iii. $E_3(\bar{N}, \bar{P}) =$

$$E_3 \left[\frac{\mu}{a}(D+N), \frac{1}{a} \left\{ (D+N) \left(1 - \frac{b}{r}\right) \left(1 - \frac{\mu}{a}(D+N)\right) \right\} \right]$$

The Jacobian matrix evaluated at E_3 is

$$J(E_3) = \begin{bmatrix} A_1^* & \mu \\ -B_1^* & 0 \end{bmatrix} \tag{16}$$

Where

$$A_1^* = \left(1 - \frac{b}{r}\right) \left(1 - \frac{2\mu}{a}(D+N)\right) - \frac{1}{D+N} \left[(D+N) \left(1 - \frac{b}{r}\right) \left(1 - \frac{\mu}{a}(D+N)\right) \right]$$

$$B_1^* = -\frac{1}{D+N} \left[(D+N) \left(1 - \frac{b}{r}\right) \left(1 - \frac{\mu}{a}(D+N)\right) \right]$$

The determinant of the matrix is found by

$$|J\left[\frac{\mu}{a}(D+N), \frac{1}{a} \left\{ (D+N) \left(1 - \frac{b}{r}\right) \left(1 - \frac{\mu}{a}(D+N)\right) \right\}\right] - \lambda I| = \begin{vmatrix} A_1^* - \lambda & \mu \\ -B_1^* & 0 - \lambda \end{vmatrix}$$

This gives $\lambda^2 - \lambda A_1^* - \mu B_1^* = 0$,

$$\lambda^2 - \left[\left(1 - \frac{b}{r}\right) \left(1 - \frac{2\mu}{a}(D+N)\right) - \frac{1}{D+N} \left[(D+N) \left(1 - \frac{b}{r}\right) \left(1 - \frac{\mu}{a}(D+N)\right) \right] \right] \lambda - \mu \frac{1}{D+N} \left[(D+N) \left(1 - \frac{b}{r}\right) \left(1 - \frac{\mu}{a}(D+N)\right) \right] = 0$$

(17)

According to the quadratic formula, the solutions are

$$\lambda = \frac{A_1^* \pm \sqrt{(-A_1^*)^2 + 4\mu B_1^*}}{2}$$

The eigen values are positive if the value generated by the radical is less than A_1^* , and with all parameter values positive, and the eigen values are negative if the value generated by the radical is more than A_1^* . Hence, we have stable equilibrium when the eigen values are negative and an unstable equilibrium when the eigen values are positive.

This is possible only when conditions (8) and (9) are put in place, that is, $b < r$ and $\mu < \frac{a}{D+N}$.

$b < r$, implies that prey's cumulative density b , which is the regulatory term, must be less than its growth rate r , whereas $\mu < \frac{a}{D+N}$, implies that, predator's mortality rate must be less than the quotient of the maximum killing rate over the sum of half-saturation constant and the prey density.

Global stability of the equilibrium points

Definition 1: Positive definite (Hendrick and Girard, 2005)

A function $V(x, y)$ which is continuously differentiable is to be positive definite in a region U that contains the origin if

- i.) $V(0, 0) = 0$
- ii.) $V(x) > 0$ and $V(y) > 0$, for all $x, y \in U - \{0\}$.

Definition 2: The Lyapunov function

A Lyapunov function $V(x,y)$ is defined as follows;

- i.) V and all its partial derivatives $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ are continuous.
- ii.) V is positive, that is, $V(0, 0) = 0$ if and only if $x = x_0, y = y_0$ and $V(x) > 0, V(y) > 0$, for all $x, y \in U - \{0\}$

A Lyapunov function $V(x, y)$ for a system is said to be

- i.) Positive definite if $V(x,y) > 0$ for all $x, y \in U - \{0\}$.
- ii.) Positive semi-definite if $V(x,y) \geq 0$ for all $x, y \in U - \{0\}$.
- iii.) Negative definite if $V(x,y) < 0$ for all $x, y \in U - \{0\}$.
- iv.) Negative semi-definite $V(x,y) \leq 0$ for all $x, y \in U - \{0\}$.

Definition 3: Lyapunov stability theorem

Let (\bar{x}, \bar{y}) be the equilibrium point of the differential equation $\dot{x} = f(x,y)$ and $V(x,y)$ be a continuously differentiable positive definite function in the neighbourhood of the origin.

- i.) If $\dot{V}(x,y) \leq 0, \forall x,y \in U - \{0\}$, then the origin is stable (Lyapunov stable).
- ii.) If $\dot{V}(x,y) < 0, \forall x,y \in U - \{0\}$, then the origin is uniformly asymptotically stable.
- iii.) If $\dot{V}(x,y) > 0, \forall x,y \in U - \{0\}$, then the origin is unstable.

Therefore a function $V(x,y)$ is a Lyapunov function if

- i.) $V(x,y) = 0$, only if $x = x_0, y = y_0$
- ii.) $V(x,y) > 0, \forall x,y \in U - \{0\}$
- iii.) $\dot{V}(x,y) < 0, \forall x,y \in U - \{0\}$

If $\dot{V}(x,y) < 0$, then $V(x,y)$ is a strict Lyapunov.

To get $\dot{V}(x,y)$, for a two dimensional system, the following formula is used:

$$\frac{dV}{dt} = \dot{V}(x,y) = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial V}{\partial x} x' + \frac{\partial V}{\partial y} y'$$

It is easy to verify that V is zero at the equilibrium and is positive for all values of x and y , from the definitions stated above.

The non-trivial global stabilities that we are going to look for are those of E_1, E_2 and E_3 .

- i. Global stability of $E_1(\bar{N}, 0) = E_1(1, 0)$

We consider a Lyapunov function as $V(N, 0) = N - \bar{N} - N \ln \left(\frac{N}{\bar{N}}\right)$.

Differentiating V with respect to time t we get,

$$\dot{V}(N, 0) = \left(\frac{N - \bar{N}}{N}\right) \dot{N}(t)$$

Substituting $\dot{N}(t)$ using equation (6) gives,

$$\dot{V}(N, 0) = (N - \bar{N}) \left[\left(1 - \frac{b}{r}\right) (1 - N) - \frac{aP}{D + N} \right]$$

For $E_1(\bar{N}, 0), P = 0$. Thus,

$$\dot{V}(N, 0) = (N - \bar{N}) \left[\left(1 - \frac{b}{r}\right) (1 - N) \right]$$

From this we get,

$$\dot{V}(N, P) = (N - \bar{N}) \left[\left(\frac{b}{r} + \bar{N} + \frac{b\bar{N}}{r}\right) - \left(\frac{b}{r} + N + \frac{bN}{r}\right) \right]$$

This simplifies to, $\dot{V}(N, P) = (N - \bar{N}) \left[1 - \frac{b}{r} \right]$.

Hence $E_1(\bar{N}, 0)$ is globally asymptotically stable.

- ii. Global stability of $E_2(0, \bar{P}) = E_2(0, 0)$

We use the Lyapunov function

$$V(0, P) = P - \bar{P} - \bar{P} \ln\left(\frac{P}{\bar{P}}\right).$$

Differentiating V with respect to time t we get,

$$\dot{V}(0, P) = \left(\frac{P - \bar{P}}{P}\right) \dot{P}(t).$$

Substituting $\dot{N}(t)$ and $\dot{P}(t)$ using equation (6) gives,

$$\dot{V}(0, P) = \left(\frac{P - \bar{P}}{P}\right) \dot{P}(t)$$

From this we get,

$$\dot{V}(0, P) = (P - \bar{P}) \left[\frac{a\bar{N}}{D + \bar{N}} - \frac{aN}{D + N} \right]$$

This simplifies to,

$$\dot{V}(0, P) = -(N - \bar{N})(P - \bar{P}) \left[\frac{a}{(D + \bar{N})(D + N)} \right]$$

Hence $E_2(0, \bar{P})$ is Lyapunov stable.

iii. Global stability of $E_3(\bar{N}, \bar{P})$

Consider the following Lyapunov function,

$$V(N, P) = N - \bar{N} - \bar{N} \ln\left(\frac{N}{\bar{N}}\right) + P - \bar{P} - \bar{P} \ln\left(\frac{P}{\bar{P}}\right)$$

Differentiating V with respect to time t , we get,

$$\dot{V}(N, P) = \left(\frac{N - \bar{N}}{N}\right) \dot{N}(t) + \left(\frac{P - \bar{P}}{P}\right) \dot{P}(t)$$

Substituting in the expression for $\dot{N}(t)$ and $\dot{P}(t)$ from equation (6), we get

$$\dot{V}(N, P) = (N - \bar{N}) \left[\left(1 - \frac{b}{r}\right)(1 - N) - \frac{aP}{D + N} \right] + (P - \bar{P}) \left[\mu - \frac{aN}{D + N} \right]$$

From this we get,

$$\dot{V}(N, P) = (N - \bar{N}) \left[\left(\frac{b}{r} + \bar{N} + \frac{b\bar{N}}{r} + \frac{a\bar{P}}{D + \bar{N}}\right) - \left(\frac{b}{r} + N + \frac{bN}{r} + \frac{aP}{D + N}\right) \right] + (P - \bar{P}) \left[\frac{aN}{D + \bar{N}} - \frac{aN}{D + N} \right]$$

This simplifies to,

$$\dot{V}(N, P) = -(N - \bar{N}) \left[1 + \frac{b}{r} \right] - (N - \bar{N})(P - \bar{P}) \left[\frac{a}{(D + \bar{N})(D + N)} \right] - (P - \bar{P}) \left[\frac{a}{(D + \bar{N})(D + N)} \right]$$

Therefore, $E_3(\bar{N}, \bar{P})$ is globally asymptotically stable.

Conclusion

In this paper, we looked at the existence of all the possible three equilibrium points. At each point, we identified conditions necessary for them to exist. It was found out that cereal aphids can exist on their own in the absence of ladybirds. However, their population would increase until it reaches the host plants carrying capacity K , then start to reduce all over again resulting into a periodic nature of population change. The ladybird cannot exist on their own

in the absence of aphids, which they feed on. The co-existence of these two species requires $b < r$ and $\mu h < \frac{a}{D + N}$. These inequalities show the parameters that must be controlled for this co-existence to occur. The conditions for local and global stability were also established. The conditions for the local stability were in most cases found to be similar to those for the existence. The conditions for the global stability states E_1, E_2 and E_3 were established by developing a suitable Lyapunov function. With the differentiated Lyapunov function, it was found that E_1 and E_3 each gave a negative value, hence were globally asymptotically stable. E_2 was Lyapunov stable. The conditions necessary for the global stability state of E_3 are similar to those for existence. For the global stability of the co-existence of the prey and the predator, the prey's death rate must be less than its growth rate. Also, the predator's mortality rate must be less than the quotient of the maximum killing rate over half the sum of the saturation rate and the population of the prey.

Conflict of Interest

Authors declare there are no conflicts of interest.

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