Math 4381/6378 Symmetry

Last class we showed

$$\frac{dy}{dx} = \frac{(xy-1)^3}{y}$$

was invariant under

$$\bar{x} = x+\varepsilon, \quad \bar{y} = \frac{y}{\varepsilon y+1}.$$

Consider the ODE

$$\frac{dy}{dx} = \frac{x+y+y^2}{xy}$$

& the Lie Group

$$\bar{x} = \frac{x}{1+\varepsilon x}, \quad \bar{y} = \frac{y}{1+\varepsilon x}.$$

Show the ODE is invariant

so $\frac{d\bar{y}}{d\bar{x}} = \dfrac{\frac{d}{dx}\left(\frac{y}{1+\varepsilon x}\right)}{\frac{d}{dx}\left(\frac{x}{1+\varepsilon x}\right)} = \dfrac{\left((1+\varepsilon x)y' - \varepsilon y\right)/\cancel{(1+\varepsilon x)^2}}{\frac{(1+\varepsilon x)1 - \varepsilon x}{(1+\varepsilon x)^2}}$

$$= (1+\varepsilon x)y' - \varepsilon y$$

Also $\dfrac{\bar{x}+\bar{y}+\bar{y}^2}{\bar{x}\bar{y}} = \dfrac{\frac{x}{1+\varepsilon x} + \frac{y}{1+\varepsilon x} + \frac{y^2}{(1+\varepsilon x)^2}}{\frac{xy}{(1+\varepsilon x)^2}}$

$$= \frac{(1+\varepsilon x)(x+y) + y^2}{xy}$$

so $\dfrac{d\bar{y}}{d\bar{x}} = \dfrac{\bar{x}+\bar{y}+\bar{y}^2}{\bar{x}\bar{y}}$

$\Rightarrow \quad (1+\varepsilon x)y' - \varepsilon y = \dfrac{(1+\varepsilon x)(x+y) + y^2}{xy}$

$$= \frac{x+y+y^2}{xy} + \frac{\varepsilon x(x+y)}{xy}$$

$$\Rightarrow (1+\varepsilon x)y' = \frac{x+y+y^2}{xy} + \frac{\varepsilon x(x+y)}{xy} + \varepsilon y$$

$$= \frac{x+y+y^2}{xy} + \frac{\varepsilon x(x+y+y^2)}{xy}$$

$$= (1+\varepsilon x)\left(\frac{x+y+y^2}{xy}\right)$$

$$\Rightarrow y' = \frac{x+y+y^2}{xy}$$ so yes invariant

Now consider the change of variables

$$x = \frac{1}{s}, \quad y = \frac{r}{s}$$

Lets see what happens to the ODE

$$\frac{dy}{dx} = \frac{\frac{d}{dr}(y)}{\frac{d}{dr}(x)} = \frac{\frac{1\cdot s - rs'}{s^2}}{-\frac{s'}{s^2}} = \frac{s-rs'}{-s'} = \frac{rs'-s}{s'}$$

so $\dfrac{dy}{dx} = \dfrac{x+y+y^2}{xy}$

$$\Rightarrow \quad \frac{rs'-s}{s'} = \frac{\frac{1}{s}+\frac{r}{s}+\frac{r^2}{s^2}}{\frac{r}{s^2}} = \frac{s+rs+r^2}{r}$$

$$\cancel{r} - \frac{\cancel{s}}{s'} = \frac{\cancel{s}}{r} + \cancel{s} + \cancel{r}$$

$$-\frac{1}{s'} = \frac{1}{r} + 1 = \frac{1+r}{r}$$

so $s' = -\dfrac{r}{1+r}$ easy to integrate

$$s = -r + \ln|1+r| + c$$

Now $x = \frac{1}{s}$, $y = \frac{r}{s}$ $\Rightarrow$ $r = y/x$, $s = \frac{1}{x}$

$$\frac{1}{x} = -\frac{y}{x} + \ln\left|1+\frac{y}{x}\right| + c \quad \text{sol}^{n}$$

Recap

$$\frac{dy}{dx} = \frac{x+y+y^2}{xy}$$

LG 1 $\bar{x} = \frac{x}{1+\varepsilon x}$, $\bar{y} = \frac{y}{1+\varepsilon x}$

ε! tx $x = \frac{1}{s}$, $y = \frac{r}{s}$

gives $\frac{ds}{dr} = -\frac{r}{1+r}$

and the new eq^n is invariant under

LG2 $\bar{r} = r$, $\bar{s} = s + \varepsilon$

Consider $x = \frac{1}{s}$, $y = \frac{r}{s}$

is this invariant under LG1 & LG2 ?

(1) $\bar{x} = \frac{1}{\bar{s}}$ so $\frac{x}{1+\varepsilon x} = \frac{1}{s+\varepsilon}$

$x(s+\varepsilon) = 1+\varepsilon x$

$xs + \cancel{\varepsilon x} = 1 + \cancel{\varepsilon x} \Rightarrow x = \frac{1}{s}$ ✓

(2) $\bar{y} = \frac{\bar{r}}{\bar{s}}$ so $\frac{y}{1+\varepsilon x} = \frac{r}{s+\varepsilon}$

$\Rightarrow \quad y(s+\varepsilon) = r(1+\varepsilon x)$

$ys + \varepsilon y = r + \varepsilon x r$

$= r + \varepsilon \frac{r}{s}$

$y(\cancel{s+\varepsilon}) = \frac{r}{s}(\cancel{s+\varepsilon})$

$y = \frac{r}{s}$ ✓

Now suppose ε gives you the LG 1

so $\bar{x} = \dfrac{x}{1+\varepsilon x}$, $\bar{y} = \dfrac{y}{1+\varepsilon x}$

& some unknown transformation

$$x = f(r,s), \quad y = g(r,s)$$

& we wish this to be invariant under LG1's

i.e. $\bar{x} = f(\bar{r},\bar{s}) \quad \bar{y} = g(\bar{r},\bar{s})$

$\Rightarrow \quad x = f(r,s) \quad y = g(r,s)$

then $\dfrac{x}{1+\varepsilon x} = f(r, s+\varepsilon)$, $\dfrac{y}{1+\varepsilon x} = g(r, s+\varepsilon)$

$$\Rightarrow \quad \frac{f(r,s)}{1+\varepsilon f(r,s)} = f(r, s+\varepsilon)$$

$$\frac{g(r,s)}{1+\varepsilon f(r,s)} = g(r, s+\varepsilon)$$

?. How do we find f & g?