Some Aspects of Laplace Autoregressive Model on Application Point of View

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Abstract- Nowadays time series models are major tools for data analysis and forecasting the various real life situations. The forecasting of time series data provides useful information to the organization which is necessary for making important decisions for government organizations as well as private organizations. The Laplace Autoregressive model is not a very new concept but a few practical, real life work has till be done up to present date. In this paper, we are providing a review on Laplace autoregressive time series model. We discussed the overall Laplace autoregressive model with various modifications for analysis of various time series data. This study gives the reader an informed about the various researches that take place within Laplace autoregressive model approach using the time series analysis.

I. INTRODUCTION

The analysis of experimental data that have been observed at different points of time leads to unique problems in statistical modelling and inference. There are many natural occurring times series observations which show a tendency to follow asymmetric and heavy tailed distributions. Various researchers discussed the application of the Laplace distribution in different areas. Time series models for real valued observation using non- Gaussian distribution has been increased from the last few decades. The era of linear time series models began with autoregressive models Autoregressive models with normal distribution used as a statistical tool for analyzing various time series data analysis. Yule and Walker introduced Autoregressive Moving Average (ARMA) model. Autoregressive Integrated Moving Average (ARIMA) model is proposed by Box and Jenkins [5] and they give detailed knowledge about ARIMA and Seasonal ARIMA models in their book. Frances H. P. & Paap [4] discussed in their book about periodic time series models. In recent times there has been considerable research in the development of time series models with seasonal or periodic properties in the meteorological and hydro logical area. In ARIMA model when the explanatory variable is added, then it was converted to ARIMAX model which is useful for analysis of multivariate time series data. Hamilton, James [9] in their book "Time Series Analysis" gives inclusive study about the vector autoregressive time series models. . Another concept of autoregressive model known as Periodic Autoregressive (PAR) model and Laplace Autoregressive (LAR) has also been frequently used recently in various environmental, hydrological and meteorological studies [25]. Gaussian process is very convenient in environmental sciences, but it does not allow for skewed marginal distribution. Gastwirth and Wolff [6] had developed a stationary linear first order autoregressive with the marginal distribution process, i.e. Called LAR (1). Gaver and Lewis [7] developed a linear AR (1) process satisfying with the gamma marginal distribution. The Laplace LAR (1) model, and its generalizations to higher order correlation structures, proposed by Dewald and Lewis [2], the autoregressive process by using Laplace distribution and is termed as Laplace Autoregressive (LAR) model. In 1989, Damsleth and El- Shaaravi [1] developed a time series model with Laplace noise as an alternative to the normal distribution. There is very little research work has been done in this area up to till date, some of them are [15, 16, 24]. Gibson J. D. [8] applied an AR (1) process for image source modeling in data compression tasks. C. H. Sim [28] discussed the general theory of model-building approach which consists if model identification, estimation, diagnostic checking and forecasting for a model with given marginal distribution.

II. LITERATURE REVIEW

- Dewald et.al. [3] Introduced *l*- Laplace process with first order autoregressive structure. These are Markov process with the geometric autocorrelation function which is typical of Gaussian, first order autoregressive process. The *l*- Laplace random variable is not infinitely divisible, but also self decomposable. They have defined new processes are called the *l*- Beta-Laplace AR (1) process, *l*- Beta –Laplace ARMA processes and *l*- Laplace ARIMA processes.
- Jaykumar K., Kalyanaraman K. And Pillai R. N. [10] had studied first order autoregressive α- Laplace process is introduced and studied as a generalization of the Laplace process by Lawrance [19] and Dewald and Lewis [2].
- Kuttykrishnan A.P. [16] had studied about the Laplace autoregressive time series models. He had also discussed the Laplace distribution as a symmetric and asymmetric. The properties of asymmetric Laplace Autoregressive model have also explained. After that, he defines the Generalized asymmetric Laplace process for first order autoregressive.
- Krishnan B. And George D. [14] defined a first order moving average model with Laplace marginal distributions extension of higher order. A first order moving average process with mixed Laplace distributions

as marginal is developed and also introduced this process as the mixture of asymmetric Laplace marginals.

- Nguyen H. D. et. al. [23] introduced the Laplace mixture autoregressive model (LMAR) model that utilizes a Laplace mixture conditions model, as an alternative to the Gaussian mixture autoregressive (GMAR) model.
- Jayakumar and Kuttykrishnan [11] developed time series models and discussed the application of asymmetric Laplace distribution in modeling currency exchange rate, interest rate, stock price changes etc.
- Seethalekshami and Jose (2004; 2006) introduced various autoregressive models utilizing α- Laplace and Pakes distributions [26, 27].
- Jose et. al [13] and Lishamol and Jose [20] developed a unified theory for a Gaussian and non- Gaussian autoregressive processes through normal-Laplace and generalized normal- Laplace distributions
- Kuttykrishnan and Jayakumar [17] introduced bivariate semi α- Laplace distribution, its characterizations and associated autoregressive models.

III. VARIOUS MODELLING TECHNIQUES FOR LAR MODEL

Various statistical tools and probabilistic mechanisms are presently explored for the purpose of analyzing time series data in various complex fields. Wold decomposition states that a stationary process can essentially be expressed as a linear combination of present and past values of a series of non-correlated sequence. The basic time series models are Autoregressive (AR), Moving Average (MA), Autoregressive Moving Average (ARMA), Autoregressive Integrated Moving Average (ARIMA) models. In this paper, we are discussing the brief description of Laplace Autoregressive Models and changes done in below points:-

A. Laplace Autoregressive Model

Kutty Krishnan A.P [16] has developed the Laplace autoregressive model with added various new changes in the basic model. In his research work shows that the LAR (1) process has “zero defect” property. The general structure of an autoregressive process of order p for AR(p) model is written as:

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + ... + \alpha_p X_{t-p} + \varepsilon_t$$

Where ε_t is white noise, i.e. $\{\varepsilon_t\} \sim N(0,\sigma^2)$ And ε_t is non-correlated with X_s four each. The first order Laplace autoregressive model is defined as follows:

Let $\{\varepsilon_t\}$ be a sequence of independent and identically distributed random variables, then the relation $X_t = \alpha X_{t-1} + \varepsilon_t, |\alpha| < 1$ with $X_0 \underline{\underline{d}} L(\alpha)$ defines a stationary autoregressive process with the Laplace marginal distribution. LAR (1) model process and established that it has similar properties as the EAR (1) model discussed by Gaver and Lewis [7]. LAR (1) model has a zero defect property which is undesirable property. Modified the LAR (1) model in such a way that the “zero defect property” is eliminated.

1. *Asymmetric Laplace Autoregressive Process*

The Laplace distribution is symmetric, it is not appropriate for modelling data with asymmetric empirical distribution. Kozubowski et. al. [18] studied the asymmetric Laplace distribution with characteristic function. We can use this modelling in various fields like mathematical finance, communication theory, environmental sciences, biology etc. The first order autoregressive process

$X_t = \alpha X_{t-1} + \varepsilon_t, 0 \le \alpha < 1, t = 1,2,....$ is stationary with asymmetric Laplace marginal distribution iff $\{\varepsilon_t\}$ is a sequence of independent and identically distributed random variables with $X_0 \underline{\underline{d}} AL(\mu,\sigma)$. Mathai introduced a class of distributions, namely generalized Laplace distribution [21]. A random variable *X* is said to be generalized asymmetric Laplace distribution if its characteristic function is given by

$$\psi(t) = \left(\frac{1}{1+\sigma^2 t^2 - i\mu t}\right)^{\tau}, -\infty < \mu < \infty, \sigma, \tau \ge 0 \text{ and}$$

symbolically we write it as $X \sim GAL(\mu,\sigma,\tau)$.

2. *Product Autoregressive Model with Log- Laplace Marginal Distribution*

The Log Laplace models used in statistical analysis of different fields such as economics, sciences etc. Kozubowski and Podgorski (2000) has reviews on the many uses of the Log–Laplace distribution. K.K. Jose and M.M.Thomas (2012) consider in their paper about Log- Laplace distribution and their multivariate extensions along with applications in time series modelling using product auto regression. They introduced Log- Laplace distribution and its properties. A random variable Y is said to have a log–Laplace distribution with parameters > 0 , $\alpha > 0$ and $\beta > 0$ (LL(δ,α,β)) by K. K. Jose its probability density function is given

$$g(y) = \frac{1}{\delta}\frac{\alpha\beta}{\alpha+\beta}\begin{cases}\left(\frac{y}{\delta}\right)^{\beta-1} & for \quad 0 < y < \delta \\ \left(\frac{\delta}{y}\right)^{\alpha+1} & for \quad y \ge \delta\end{cases}$$

This distribution can be derived by combining the two power laws and has power laws and has power tails at zero and at infinity. Kozubowski and Podgorski (2003) studied some important properties of LL (δ,α,β). It has Pareto-type tails at the zeros and infinity. The estimation of parameters of the log-Laplace distribution is given by Hinkley and Revankar (1977). In this paper author had also discussed the multivariate extension and divisibility properties of Log- Laplace distribution, then define a product autoregressive structure which introduced by Mckenzie and the self decomposability property is studied. A product auto regression structure of order one (PAR (1)) has the form

$$Y_n = Y_{n-1}^{a}\varepsilon_n,\ 0 < a \le 1, n = 0 \pm 1, \pm 2, \ldots\ldots, \ldots$$

Where $\{\varepsilon_n\}$ is a sequence of i.i.d. positive random variables. When we take logarithms of Y_n and then let $X_n = \log Y_n$, then the stationary process of $\{X_n\}$ has the form $X_n = aX_{n-1} + \eta_n,$ Where $\eta_n = \log \varepsilon_n$,

A linear AR (1) model was developed along with the sample path properties and the estimation of parameters of the process. Also consider the multivariate extension of the product auto regression structure has also considered.

B. α -Laplace Process

The first order autoregressive α -Laplace Process is introduced and studied as a generalization of the Laplace process of Lawrence (1978) and Dewald and Lewis (1985). K. Jayakumar, K. Kalyanaraman and R. N. Pillai (1995) have introduced α -Laplace Process. The autoregressive α -Laplace model has been built by using a sequence of independent and identically distributed α -Laplace random variables in below equation.

$$X_n = \begin{cases} bX_{n-1} & with\ \ probability \quad b^{\alpha} \\ bX_{n-1} + e_n & with\ \ probability \quad (1 - b^{\alpha}) \end{cases}$$

$0<b<1$, $0<\alpha \le 2$. The first- order autoregressive nature of the process is evident with b and α are parameters. The simulated sample path is also discussed in this paper.

C. l- Laplace process

Dewald, Lewis and Mckenzie (1995) introduced *l*- Laplace family of distribution and make *l*-Laplace process with the first order autoregressive model. There are Markov processes with the geometric autocorrelation function which is typical of the Gaussian, which defined in first part *l*- Beta-Laplace autoregressive model. In second point they introduced encompassing *l*-Beta ARMA autoregressive moving average process. A brief introduction is given below:-

1. l-Beta –Laplace first order Autoregressive Model

The *l*- Laplace random variable has infinitely divisible and self decomposable. Therefore a linear, constant coefficient auto regression can be defined. The stationary process of $\{X_t(l)\}$ by means of an additive, random coefficient equation:

$$X_t(l) = M_t^{1/2}(l\alpha, l\overline{\alpha})X_{t-1}(l) + N_t^{1/2}(l\overline{\alpha}, l\alpha)L_t(l)$$

Where $\{M_t(l\alpha, l\overline{\alpha})\}$ is an i.i.d. sequence of Beta $(l\alpha, l\overline{\alpha})$ random variables; $\{N_t(l\overline{\alpha}, l\alpha)\}$ is an i.i.d., sequence of Beta $(l\overline{\alpha}, l\alpha)$ random variables, independent of $\{M_t(l\alpha, l\overline{\alpha})\}$ and $L_t(l)$ is an i.i.d. sequence, independent of both coefficient sequences of *l*-Laplace random variables. This process is called *l- Beta- Laplace* AR(1) and also check the sample path behavior for *l- Beta –Laplace* processes for fixed lag one serial correlation and different value of *l*.

2. The l-Beta –Laplace Moving average process

In this process, we define $\{L_t(\theta)\}$ be an i.i.d. sequence of θ -Laplace random variables, where $\theta = l/(1+\beta)$ and $0<\beta<1$ and let $\{C_t(\theta\beta, \overline{\theta\beta})\}$ be an i.i.d. sequence of Beta $(\theta\beta, \overline{\theta\beta})$ varietes independent of $\{L_t(\theta)\}$. Then the process $\{X_t(l)\}$ generated by

$$X_t(l) = L_t(\theta) + C_t^{1/2}(\theta\beta, \overline{\theta\beta})L_{t-1}(\theta)$$

has a marginal *l*- Laplace distribution and an MA(1) structure with $0 \le Corr(X_n, X_{n-1}) \le 0.5$. The above equation of $X_t(l)$ is a random coefficient moving average of order one.

3. The l-Beta Laplace Autoregressive Moving average process

The mixed model is obtained from combining process of autoregressive component and the other is moving average components, then obtained the first order *l*-Beta Laplace Autoregressive Moving Average (*l*-Beta LARMA) which is obtained by defining the process $\{X_t(l)\}$ by means. Firstly, we show that $X_t(l)$ is marginally *l*-Laplace and has the autocorrelation function of an ARMA (1,1) process. In particular

$$\rho_X(k) = \tau(\theta\beta)\frac{[\tau(\theta\overline{\alpha}) + \tau(\theta\beta) + \tau(\theta\alpha)][\tau(\theta\alpha)]^{k-1}}{(1+\beta)},$$

for k= 1,2,…… This process $\{X_t(l)\}$ has a structure which is determined in a simple way by the two parameters α *and* β.

C. Laplace Mixture Autoregressive Models

The standard AR model allows for the unimodal marginal and conditional densities, these models do not capture conditional heteroscedasticity. To alleviate this drawback, Wong and Li [29] proposed the Gaussian Mixture Autoregressive (GMAR) model. Nguyen et.al. [23] proposed the Laplace Mixture Autoregressive (LMAR) model. Let Y_t arises from a *g* component LMAR model of order *p* i.e. an LMAR (*g*, *p*) model if $Y_t \mid F_{t-1}$ has a density of the form

$$f\left(y_t \mid F_{t-1};\theta\right) = \sum_{i=1}^{g} \pi_i \lambda(y_t;\beta_{i0} + \sum_{j=1}^{p}\beta_{ij} y_{t-j}, \xi_i)$$

Where π_i and β_i are the same as in (1) and $\xi > 0$ for all i=1,..,g. Here

$\lambda(y;\mu,\xi) = (\xi\sqrt{2})^{-1}\exp(-\sqrt{2}|x-\mu|/\xi)$ is the Laplace density function with mean μ and variance ξ^2, and $\theta = (\pi_1,....,\pi_{g-1},\beta_1^T,...,\beta_g^T,\xi_1,......,\xi_g)^T$ is the model parameter vector. They also provide some characterization of the LMAR model and discuss some stationary conditions; gives demonstrate the use of the LMAR model. Consider the analysis of a time series data set arising from the calcium imaging of a zebra fish brain.

D. Mixed Asymmetric Laplace Moving Process

The Laplace and asymmetric Laplace distribution are considered a good choice for modelling whenever the data has heavier tails than Gaussian tails. Krishnan B. and George D. [14] discussed in their paper about the Laplace moving average process, its extension and proposed mixed asymmetric Laplace moving average process . Let $\{X_n\}$ be a sequence of random variables with structure

$$X_n = \begin{cases} \varepsilon_n & w.p \quad p_0 \\ \beta\varepsilon_{n-1} + \varepsilon_n & w.p \quad p_1 \end{cases}$$

Where $0 \le \beta \le 1; 0 \le p_0; p_1 \le 1$ and $p_0 + p_1 = 1$

IV. APPLICATIONS

A lot of theoretical work on Laplace distribution is found recently in the literature of various fields like time series modelling, financial modelling, communication engineering, image source modelling, gene expression data modelling etc. which is a good alternative of normal distribution with applications in time series modelling. Damsleth E. and Saharawi A. H. [1] applied ARMA model with double exponentially distributed noise on experimental data of weekly measurements of sulphate concentration. Sim C. H. [28] had applied different type of autoregressive models with various distribution in his paper on the discharge of the Mekong river. In recent years Johnson et. al. [12] applied Laplace distribution models in their paper on data related to road topography and roughness. Miftahurromah B. [22] et. al. used Bayesian mixture Laplace Autoregressive approach for modelling the Islamic stock risk investment. The Laplace distributions consider for modelling whenever data exhibiting heavier tail, the models discussed in this paper could be the better models for various time series data fitting.

V. CONCLUSION

The interest on developing time series modelling for real valued observation using non-Gaussian distribution has increased tremendously from last two decades. The necessity of real valued and natural occurring time series observation yields the interest in this field using a Laplace distribution with autoregressive process. Various authors discussed the application of Laplace autoregressive process in different fields that from communication theory to environmental sciences. In this paper our purpose is to provide the important information.

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