Math 6345 Adv. ODE's

Consider

$$\frac{d\bar{x}}{dt} = a(t)\,\bar{x}$$

where $a(t+T) = a(t)$ T-periodic

Q 1. Is the sol^n periodic?

2. Is the sol^n T-periodic?

Does the sol^n have any structure whatsoever?

Consider $\frac{dx}{dt} = \cos t\, x$

so here $a(t) = \cos t$ & $a(t+2\pi) = \cos(t+2\pi)$
$= \cos t$
$= a(t)$

so $a(t)$ is 2π-periodic.

we can solve this

$$\frac{dx}{x} = \cos t\, dt$$

$$\ln x = \sin t + \ln c$$

$$\Rightarrow \quad x(t) = c\, e^{\sin t}$$

and $\quad x(t+2\pi) = c\, e^{\sin(t+2\pi)} = c\, e^{\sin t} = x(t)$

so in this ex. yes,

Consider $\quad \dfrac{dx}{dt} = (\cos t + 1)\, x$

$a(t) = \cos t + 1 \quad \&\quad a(t+2\pi) = a(t) \quad$ again

however $\quad \dfrac{dx}{x} = (\cos t + 1)\, dt$

$$\ln x = \sin t + t + \ln c$$

$$x(t) = c\, e^{\sin t + t} = c\, e^{\sin t}\, e^{t}$$

$$x(t+2\pi) = c\, e^{\sin(t+2\pi)}\, e^{t+2\pi} = c\, e^{\sin t} \cdot e^{t} \cdot e^{2\pi}$$

so $x(t+2\pi) = x(t)\, e^{2\pi}$

and so, in this example, no, the sol^n is not periodic.

This sol^n does have structure though

$$x(t) = \underbrace{c\,e^{\sin t}}_{\text{periodic}} - \underbrace{e^{t}}_{\text{exponential term}}$$

<u>Ex2</u> $\quad \dfrac{d\bar{x}}{dt} = \begin{pmatrix} 1 & \cos t \\ 0 & 1 \end{pmatrix} \bar{x} \qquad A(t) = \begin{pmatrix} 1 & \cos t \\ 0 & 1 \end{pmatrix}$

$A(t+2\pi) = A(t)$

so $\dfrac{dx}{dt} = x + \cos t\, y \qquad \dfrac{dy}{dt} = y \Rightarrow y = c_1 e^{t}$

so $\dfrac{dx}{dt} - x = c_1 e^{t} \cos t$

$\mu = e^{-t} \Rightarrow \dfrac{d}{dt}\left(e^{-t} x\right) = c_1 \cos t$

$\Rightarrow \quad e^{-t} x = c_1 \sin t + c_2$

$\Rightarrow \quad x = c_1 e^{t} \sin t + c_2 e^{t}$

so $\bar{x} = \begin{pmatrix} e^t \sin t & e^t \\ e^t & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Is there structure here?

<u>Note:</u> $\phi(t) = \begin{pmatrix} e^t \sin t & e^t \\ e^t & 0 \end{pmatrix}$

$$\phi(t+2\pi) = \begin{pmatrix} e^{t+2\pi} \sin(t+2\pi) & e^{t+2\pi} \\ e^{t+2\pi} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{t+2\pi} \sin t & e^{t+2\pi} \\ e^{t+2\pi} & 0 \end{pmatrix} \neq \phi(t)$$

but

$$= \begin{pmatrix} e^t \sin t & e^t \\ e^t & 0 \end{pmatrix} \begin{pmatrix} e^{2\pi} & 0 \\ 0 & e^{2\pi} \end{pmatrix}$$

$= \phi(t) \; * \nearrow$ this constant matrix so does the sol^n ~~have~~ structure. Yes!

Note:

$$\Phi(t) = \begin{pmatrix} e^t \sin t & e^t \\ e^t & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin t & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$$

let $P(t) = \begin{pmatrix} \sin t & 1 \\ 1 & 0 \end{pmatrix}$

with $P(t+2\pi) = P(t)$

Further if $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

then $\begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} = e^{Bt}$

so $\Phi(t) = P(t)\, e^{Bt}$

Floquet Th^y

if $\frac{d\bar{x}}{dt} = A(t)\,\bar{x}$

when $A(t+T) = A(t)$

and $\Phi(t)$ is the fundemental matrix

(1) $\Phi(t+T) = \Phi(t)\,C$ for some constant matrix C

(2) $\Phi(t) = P(t)\,e^{Bt}$

when $P(t+T) = P(t)$ B constant Matrix

(3) $C = e^{BT}$

(4) under $\bar{x} = P(t)\,\bar{u}$

system becomes

$$\dot{\bar{u}} = B\,\bar{u}$$

Let's show the last part

$$\bar{x} = \begin{pmatrix} \sin t & 1 \\ 1 & 0 \end{pmatrix} \bar{u}$$

so $x = \sin t\, u + \cdot\, v$

$y = u$

$$\frac{dx}{dt} = x + \cos t\, y \Rightarrow \sin t\, \dot{u} + \cancel{\cos t\, u} + \dot{v}$$
$$= \sin t\, u + v + \cancel{\cos t}\, u$$

$$\Rightarrow \sin t\, (\dot{u} - u) + \dot{v} + v = 0$$

$$\frac{dy}{dt} = y \Rightarrow \dot{u} = u \quad \text{so} \quad \dot{v} = v$$

and in this example

$$\dot{\bar{u}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{u}$$

$\uparrow$

and this was B.