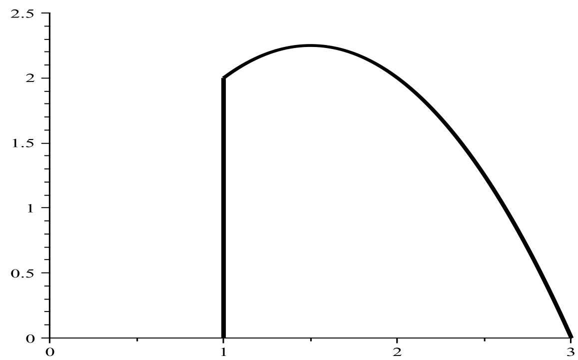
Math 1496 - Sample Test 3 - Solns

1. Using n rectangles and the limit process, find the area under the given curve.

$$y = 3x - x^2 \text{ on } [1,3]$$



Sol: The thickness of each rectangle is $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$. We choose $x_i = 1 + \dfrac{2i}{n}$ so the height of the i^{th} rectangles is $h_i = f(x_i) = 3\left(1+\dfrac{2i}{n}\right) - \left(1+\dfrac{2i}{n}\right)^2$. Next, the area of this rectangle is $A_i = f(x_i)\Delta x = \left[3\left(1+\dfrac{2i}{n}\right) - \left(1+\dfrac{2i}{n}\right)^2\right]\dfrac{2}{n}$

Thus,

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(1+\frac{2i}{n}\right) - \left(1+\frac{2i}{n}\right)^2\right]\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n} + \frac{4i}{n^2} - \frac{8i^2}{n^3}\right) \\
&= \lim_{n\to\infty} \left(\frac{4}{n}\cdot n + \frac{4}{n^2}\cdot\frac{n(n+1)}{2} - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\
&= 4 + 2 - \frac{8}{3} = \frac{10}{3}
\end{aligned}$$

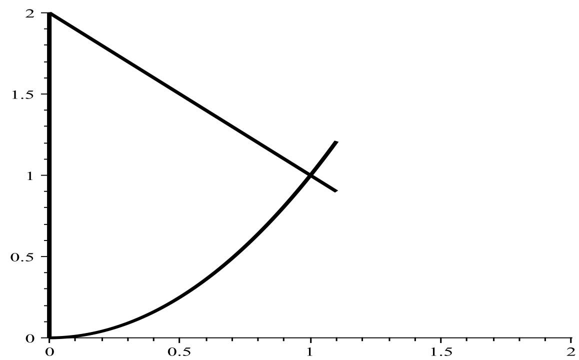
2. Find the area bound by the following curves

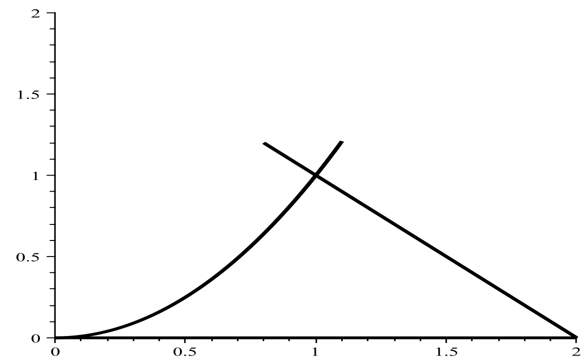
$$y = x^2 \quad y = 2 - x, \quad x = 0, \quad x, y \geq 0.$$

We sketch the curves to find the region of interest (the one on the left). The intersection points between the two curves are

$$x^2 = 2 - x \;\Rightarrow\; x^2 + x - 2 = 0 \;\Rightarrow\; (x+2)(x-1) = 0 \;\Rightarrow\; x = 1, -2$$

and only $x = 1$ is applicable.





The area is then given by

$$A = \int_0^1 \left(2 - x - x^2\right) dx = 2x - \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 = 2 - \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$$

Often one mistakes the region and calculates the other region (the one one the right) so we'll do it here. Using vertical rectangles, we'll need two integrals so

$$\begin{aligned} A &= \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx \\ &= \frac{x^3}{3}\bigg|_0^1 + \left(2x - \frac{x^2}{2}\right)\bigg|_1^2 \\ &= 1/3 + ((4-2) - (2 - 1/2)) = 5/6 \end{aligned}$$

Using horizontal rectangle we note the intersection point of $x = 1$ which gives $y = 1$ and so the area is

$$A = \int_0^1 (2 - y - \sqrt{y})\ dy = 2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\bigg|_0^1 = 5/6$$

3. Evaluate the following

$$(i)\ \frac{d}{dx}\int_1^x \sin\left(t^2\right)\ dt = \sin\left(x^2\right)$$

$$\begin{aligned} (ii)\ \frac{d}{dx}\int_x^{x^2} \sqrt{1+t^2}\,dt &= \frac{d}{dx}\int_x^0 \sqrt{1+t^2}\,dt + \frac{d}{dx}\int_0^{x^2} \sqrt{1+t^2}\,dt \\ &= -\frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt + \frac{d}{dx}\int_0^{x^2} \sqrt{1+t^2}\,dt \\ &= -\sqrt{1+x^2} + \sqrt{1+(x^2)^2}\cdot 2x \end{aligned}$$

4. Find the following limits

$(i)\ \lim\limits_{x\to\infty} \dfrac{e^x - 1}{e^x + 1}$

Soln: Applying the limit we see the form $\dfrac{``\infty"}{\infty}$ so using L'H we get

$$\lim_{x\to\infty} \frac{e^x}{e^x} = 1$$

(ii) $\lim_{x\to 0^+} x\ln x$

Soln: Applying the limit we see the form "$0\cdot\infty$" so we must put the limit in proper form. Here we consider

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \frac{\text{“}\infty\text{”}}{\infty} \text{ so L'H gives } \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = -\lim_{x\to 0^+} x = 0$$

and thus

$$\lim_{x\to 0^+} x\ln x = 0$$

(iii) $\lim_{x\to 0^+} x^x$

Soln: Applying the limit we see the form "0^0" so we must put the limit in proper form. Here we consider

$$x^x = e^{\ln x^x} = e^{xlnx}$$

Since the limit of

$$\lim_{x\to 0^+} x\ln x = 0 \text{ (from (i)) then}$$

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

5. Evaluate the following indefinite integrals

(i) $\int \sec^2 x\tan x\,dx$

Let $u = \sec x$ so $du = \sec x\tan x\,dx$ and the integral becomes

$$\int u\,du = \frac{u^2}{2} + c = \frac{\sec^2 x}{2} + c$$

(ii) $\int \frac{e^{1/x}}{x^2}\,dx$

Let $u = \frac{1}{x}$ so $du = -\frac{1}{x^2}\,dx$ and the integral becomes

$$\int -e^u\,du = -e^u + c = -e^{1/x} + c$$

(iii) $\int \frac{x}{x^2+1}\,dx$

Let $u = x^2+1$ so $du = 2xdx$ and the integral becomes

$$\frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+1| + c$$

(iv) $\int_1^5 x\sqrt{x-1}\,dx$

Let $u = x - 1$ so $du = dx$ and the limits

$$x = 1 \ \Rightarrow \ u = 0 \quad \text{and} \quad x = 5 \ \Rightarrow \ u = 4$$

and the integral becomes

$$\int_0^4 (u+1)\sqrt{u}\, du = \int_0^4 u^{3/2} + u^{1/2}\, du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\bigg|_0^4 = \frac{64}{5} + \frac{16}{3} = \frac{272}{15}$$

(v) $\displaystyle\int_0^{\pi/4} \sin x \cos x\, dx$

Let $u = \sin x$ so $du = \cos x\, dx$ and the limits

$$x = 0 \ \Rightarrow \ u = 0 \quad \text{and} \quad x = \pi/4 \ \Rightarrow \ u = \sqrt{2}/2$$

and the integral becomes

$$\int_0^{\sqrt{2}/2} u\, du = \frac{u^2}{2}\bigg|_0^{\sqrt{2/2}} = \frac{1}{4}$$

(vi) $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\, dx$

Let $x = 2u$ so $dx = 2du$ and the limits

$$x = 0 \ \Rightarrow \ u = 0 \quad \text{and} \quad x = 1 \ \Rightarrow \ u = 1/2$$

and the integral becomes

$$\frac{2}{2}\int_0^{1/2} \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u\,\bigg|_0^{1/2} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}.$$