Structural Robustness to Noise in Consensus Networks: Impact of Average Degrees and Average Distances

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***Abstract*— We investigate how the graph topology influences the robustness to noise in undirected linear consensus networks. We consider the expected steady state population variance of states as the measure of vulnerability to noise. We quantify the structural robustness of a network by using the smallest value this measure can attain under edge weights from the unit interval. Our main result shows that the average distance between nodes and the average node degree define tight upper and lower bounds on the structural robustness. Using these bounds, we characterize the networks with different types of robustness scaling. We also present a fundamental trade-off between the structural robustness and the sparsity of networks. We then show that random regular graphs typically have near-optimal structural robustness among the graphs with same size and average degree. Some simulation results are also provided.**

I. Introduction

Consensus networks, where the state of each node approaches a weighted average of the states of adjacent nodes, are used to model the diffusive couplings in a variety of natural and engineered systems such as biological systems, financial networks, social networks, communication systems, transportation systems, power grids, sensor networks, and robotic swarms. These systems typically operate in the face of various disturbances such as measurement/process noise, communication delays, component failures, misbehaving nodes, or malicious attacks. Accordingly, a central question regarding such networks is how well they behave in the face of disturbances. This question has motivated many studies on the robustness of consensus networks. Graph measures such as connectivity, expansion ratios, centrality, and Kirchoff index have been used in the literature to quantify the robustness to various disturbances (e.g., [1], [2], [3], [4], [5], [6]).

This paper is focused on the robustness of undirected consensus networks to noisy interactions. In such networks, each edge is endowed with some positive weight denoting the coupling strength between the corresponding nodes. We consider a setting with additive process noise, where the state of each node is attracted towards the weighted average of the states of its neighbors plus some independent and identically distributed (i.i.d.) white Gaussian noise with zero mean and unit covariance. We use the expected steady state population variance of states, which is a variant of the $\mathcal{H}_2$-norm of the system with the output defined as the deviation of nodes from global average, as the measure of vulnerability to noise. We define the notion of structural robustness to noise, which assesses each network based on the smallest value of expected steady state variance that can be attained under the noisy consensus dynamics with edge weights from the unit interval. As such, the structural robustness is determined by the amount of vulnerability to noise, which persists even under the best allocation of edge weights, due to the network topology. We show that two simple graph measures, namely the average distance between nodes and the average node degree, define tight bounds on the proposed measure of structural robustness. We then use these bounds to obtain some fundamental graph topological limitations on structural robustness and characterize graphs with extremal robustness scaling. We also show that random k-regular graphs, which are graphs that are selected uniformly at random from the set of all graphs with n nodes such that the number of edges incident to each node (degree) is equal to k, typically have near-optimal structural robustness among the graphs of size n and average degree k.

The organization of this paper is as follows: Section II provides some graph theory preliminaries. Section III presents our main results. Section IV provides some simulation results. Finally, Section V concludes the paper.

II. Preliminaries

A. Notation

We use $\mathbb{R}$ and $\mathbb{R}_+$ to denote the set of real numbers and positive real numbers, respectively. For any finite set A with cardinality $|A|$, we use $\mathbb{R}^{|A|}$ (or $\mathbb{R}_+^{|A|}$) to denote the space of real-valued (or positive-real-valued) $|A|-$dimensional vectors. For any pair of vectors $x, y \in \mathbb{R}^{|A|}$, we use $x \leq y$ (or $x < y$) to denote the element-wise inequalities, i.e., $x_i \leq y_i$ (or $x_i < y_i$) for all $i = 1, 2, \ldots, |A|$. The all-ones and all-zeros vectors, their sizes being clear from the context, will be denoted by $\mathbf{1}$ and $\mathbf{0}$.

B. Graph Theory Basics

A graph $\mathcal{G} = (V, E)$ consists of a node set $V = \{1, 2, \ldots, n\}$ and an edge set $E \subseteq V \times V$. For an undirected graph, each edge is represented as an unordered pair of nodes. For each $i \in V$, let $\mathcal{N}_i$ denote the *neighborhood* of i, i.e.,

$$\mathcal{N}_i = \{j \in V \mid (i,j) \in E\}$$

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A path between a pair of nodes $i, j \in V$ is a sequence of nodes $\{i, \ldots, j\}$ such that each pair of consecutive nodes are linked by an edge. For any node i, the number of nodes in its neighborhood, $|\mathcal{N}_i|$, is called its degree, d_i. Accordingly, the average node degree of a graph is

$$\tilde{d}(\mathcal{G}) = \frac{1}{n} \sum_{i=1}^{n} d_i.$$

The *distance* between any two nodes i and j, which is denoted by δ_{ij}, is equal to the number of edges on the shortest path between those nodes. The maximum distance between any two nodes, $\max_{i,j \in V} \delta_{i,j}$ is known as the *diameter* of the graph, and the *average distance* between nodes is given as

$$\tilde{\delta}(\mathcal{G}) = \frac{2}{n^2 - n} \sum_{1 \le i < j \le n} \delta_{ij}.$$

A graph is *connected* if there exists a path between every pair of nodes. For weighted graphs, we use $w \in \mathbb{R}_+^{|E|}$ to denote the vector of edge weights and $w_{ij} \in \mathbb{R}_+$ to denote the weight of the edge $(i, j) \in E$. The *adjacency matrix*, $\mathcal{A}$, of a weighted graph is defined as

$$[\mathcal{A}_w]_{ij} = \begin{cases} w_{ij} & \text{if } (i,j) \in E \\ 0 & \text{otherwise,} \end{cases}$$

and the corresponding (weighted) *graph Laplacian* is

$$[L_w]_{ij} = \begin{cases} \sum_{k \in \mathcal{N}_i} \mathcal{A}_{ik} & \text{if } i = j \\ -\mathcal{A}_{ij} & \text{otherwise,} \end{cases}$$

In the remainder of the paper, we will use L to denote the unweighted Laplacian, i.e., the special case when $w = \mathbf{1}$.

C. Consensus Networks

Consensus networks can be represented as a graph, where the nodes correspond to the agents, and the weighted edges exist between the agents that are coupled through local interactions. For such a network $\mathcal{G} = (V, E)$, let the dynamics of each agent $i \in V$ be

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} w_{ij}(x_j(t) - x_i(t)) + \xi_i(t),$$

where $x_i(t) \in \mathbb{R}$ denotes the state of i, each $w_{ij} \in \mathbb{R}_+$ is a constant weight representing the strength of the coupling between i and j, and $\xi(t) \in \mathbb{R}^n$ is i.i.d. white Gaussian noise with zero mean and unit covariance, which is one of the standard noise models for agents that are independently affected by disturbances of same intensity due to various effects such as communication errors, noisy measurements, or quantization errors (e.g.,[4], [5], [7]). Accordingly, the overall dynamics of the agents can be expressed as

$$\dot{x}(t) = -L_w x(t) + \xi(t), \tag{1}$$

where L_w denotes the weighted Laplacian. In a noise-free setting ($\xi(t) = \mathbf{0}$ for all $t \ge 0$), the dynamics in (1) are known to result in a global consensus, $\lim_{t\to\infty} x(t) \in span\{\mathbf{1}\}$, for any $x(0) \in \mathbb{R}^n$ if and only if the graph is connected [8]. In the noisy case, a perfect consensus can not be achieved. Instead, some finite steady state variance of $x(t)$ is observed on connected graphs [4], [5]. Accordingly, the robustness of the network can be quantified through the expected population variance in steady state, i.e.,

$$\mathcal{H}(\mathcal{G}, w) \coloneqq \lim_{t\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}[(x_i(t) - \tilde{x}(t))^2],$$

where $\tilde{x}(t) \in \mathbb{R}$ denotes the average of $x_1(t), x_2(t), \ldots, x_n(t)$.

It can be shown that (e.g., see [4], [5]) $\mathcal{H}(\mathcal{G}, w)$ is equal to $1/n$ times the square of the $\mathcal{H}_2$-norm of the system in (1) from the input $\xi(t)$ to the output $y(t) \in \mathbb{R}^n$ defined as $y_i(t) = x_i(t) - \tilde{x}(t)$, and it satisfies

$$\mathcal{H}(\mathcal{G}, w) = \frac{1}{2n} \sum_{i=2}^{n} \frac{1}{\lambda_i(L_w)}, \tag{2}$$

where and $0 < \lambda_2(L_w) \le \ldots \le \lambda_n(L_w)$ denote the eigenvalues of the weighted Laplacian L_w.

In this paper, we investigate how much the structure of the underlying graph (the edge set E) causes vulnerability to noise in consensus networks. We measure the structural vulnerability of any given network to noise based on the smallest possible value of $\mathcal{H}(\mathcal{G}, w)$, given that the edge weights should belong to the feasible set $\mathcal{W} = \{w \mid \mathbf{0} < w \le \mathbf{1}\}$. Since multiplying all the weights by some $\alpha \in \mathbb{R}_+$ results in $L_{\alpha w} = \alpha L_w$ and $\mathcal{H}(\mathcal{G}, \alpha w) = \mathcal{H}(\mathcal{G}, w)/\alpha$ due to (2), it is possible to make $\mathcal{H}(\mathcal{G}, w)$ arbitrarily small for any network by just scaling up all the weights. By considering only weights in $(0, 1]$, we remove this possibility and focus on the impact of network structure.

Definition (Structural Vulnerability and Robustness) The structural vulnerability of an undirected consensus network $\mathcal{G} = (V, E)$ to noise is the smallest possible value of $\mathcal{H}(\mathcal{G}, w)$ that is achievable under weights from the unit interval, i.e.,

$$\mathcal{H}^*(\mathcal{G}) \coloneqq \min_{\mathbf{0} < w \le \mathbf{1}} \mathcal{H}(\mathcal{G}, w). \tag{3}$$

The structural robustness to noise is quantified via the reciprocal of structural vulnerability, $1/\mathcal{H}^*(\mathcal{G})$.

In the remainder of this paper, for brevity we will say "structural robustness (or vulnerability)" without explicitly saying "to noise". The term "structural robustness" is also used in the literature to refer to the robustness of a network's connectivity to node or edge failures (e.g., [9], [6]). While the two notions of robustness have some connections (e.g., [6], [2]), the distinction should be clear from the context.

III. Main Results

In this section, we first express stuctural vulnerability in terms of the Laplacian eigenvalues. Then, we derive tight bounds on structural vulnerability based on the average node degrees and average distances. We use these bounds to characterize graphs with extremal robustness scaling. We then show that there is a fundamental trade-off between sparsity and structural robustness. Finally, we show that random regular graphs typically have near-optimal structural robustness among the graphs of same size and sparsity.

A. Connection to the Laplacian eigenvalues

Lemma 3.1. *For any connected undirected graph $\mathcal{G}$,*

$$\mathcal{H}^*(\mathcal{G}) = \frac{1}{2n}\sum_{i=2}^{n}\frac{1}{\lambda_i(L)}, \tag{4}$$

where L denotes the unweighted Laplacian of $\mathcal{G}$.

Proof: For any connected undirected graph $\mathcal{G}$, any weighted Laplacian is a positive semidefinite matrix. Increasing any of its weights or adding new edges leads to a new Laplacian that is equal to the initial Laplacian plus another matrix that is also a weighted Laplacian (a graph with only the added/strengthened edges). All the Laplacian eigenvalues monotonically (not necessarily strictly) increases under such an addition of a positive semidefinite matrix due to the Weyl's inequality (e.g., see [10]). Hence, $\mathcal{H}(\mathcal{G}, w)$ is minimized for $w = \mathbf{1}$ within the feasible set of (3). Accordingly, using (2), we obtain (4). □

In light of Lemma 3.1, $\mathcal{H}^*(\mathcal{G})$ of any connected network can be computed through the eigenvalues of the unweighted Laplacian. Furthermore, using this result, $\mathcal{H}^*(\mathcal{G})$ can also be expressed in terms of a graph measure known as the Kirchhoff index (total effective resistance) [11], which satisfies

$$K_f(\mathcal{G}) = n\sum_{i=2}^{n}\frac{1}{\lambda_i(L)},$$

where L is the Laplacian of $\mathcal{G}$. Accordingly, due to (4),

$$H^*(\mathcal{G}) = \frac{K_f(\mathcal{G})}{2n^2}. \tag{5}$$

The connection in (5) is particularly useful as it links the structural robustness to the rich literature in graph theory on Kirchhoff index. For instance, closed form expressions in terms of size are known for some graph families (e.g., see [12], [13], [14]). Using those results on Kirchhoff index we immediately obtain that the path ($\mathcal{P}_n$), cycle ($\mathcal{C}_n$), star ($\mathcal{S}_n$), and complete ($\mathcal{K}_n$) graphs of size n have

$$H^*(\mathcal{P}_n) = \frac{n^2-1}{12n},\; H^*(\mathcal{C}_n) = \frac{n^2-1}{24n}, \tag{6}$$

$$H^*(\mathcal{S}_n) = \frac{(n-1)^2}{2n^2},\; H^*(\mathcal{K}_n) = \frac{(n-1)}{2n^2}. \tag{7}$$

Furthermore, among all the connected undirected graphs with n nodes, the Kirchoff index is minimized in the complete graph $\mathcal{K}_n$ and maximized in the path graph $\mathcal{P}_n$ (e.g., see [14]). As such, in light of (5), $\mathcal{K}_n$ and $\mathcal{P}_n$ are also the minimizer and maximizer of $H^*(\mathcal{G})$, respectively.

B. Impact of Average Degree and Average Distance

The structural vulnerability of any given network can be computed by using the Laplacian eigenvalues as in (4). Furthermore, the connection with the Kirchhoff index in (5) enables the identification of extremal graphs (path and complete) and provides closed form expressions in terms of network size for several graph families. However, it is not easy to use (4) or (5) for certain analysis and design applications in a systematic and efficient way. For instance, finding an optimal way to add a given number of edges to an arbitrary network to reduce the $\mathcal{H}^*(\mathcal{G})$ would require searching among all possibilities (e.g., see [14]). Furthermore, while it is possible to see how $\mathcal{H}^*(\mathcal{G})$ scales with size for the special graph families with closed form expressions, it is hard to do this for generic structures. One way to overcome these type of difficulties is focusing on some upper/lower bounds on $\mathcal{H}^*(\mathcal{G})$ rather than its exact value.

Many upper and lower bounds on the Kirchhoff index have been proposed in the literature by using graph measures such as chromatic number, independence number, edge/node connectivity, diameter, or degree distribution (e.g., see [7], [15], [16]). These bounds typically require significant amount of global information and/or computation, which limits their applicability in large networks (e.g., see [17], [18]). Motivated by such limitations, we present a fundamental relationship between the $\mathcal{H}^*(\mathcal{G})$ and two aggregate measures, namely the average node degree and the average distance between nodes, which can be computed/estimated efficiently (e.g. in time sublinear in network size [19]), possibly in a distributed manner with partial information (e.g., [20]). Specifically, our next result shows that these two aggregate measures define tight upper and lower bounds on $\mathcal{H}^*(\mathcal{G})$.

Theorem 3.2. *For any connected undirected graph $\mathcal{G} = (V, E)$ with $n \geq 2$ nodes,*

$$\frac{(n-1)^2}{2\tilde{d}(\mathcal{G})n^2} \leq \mathcal{H}^*(\mathcal{G}) \leq \frac{\tilde{\delta}(\mathcal{G})(n-1)}{4n}, \tag{8}$$

where $\tilde{d}(\mathcal{G})$ is the average node degree, $\tilde{\delta}(\mathcal{G})$ is the average distance between nodes. Moreover, the lower bound holds with equality if and only if $\mathcal{G}$ is a complete graph, and the upper bound holds with equality if and only if $\mathcal{G}$ is a tree.

Proof: (Lower bound:) Since the harmonic mean is always less than or equal to the arithmetic mean, we have

$$\frac{n-1}{\sum_{i=2}^{n}\lambda_i(L)} \leq \frac{1}{n-1}\sum_{i=2}^{n}\frac{1}{\lambda_i(L)}, \tag{9}$$

where the left side is the harmonic mean and the right side is the arithmetic mean of $1/\lambda_2(L), 1/\lambda_3(L), \ldots, 1/\lambda_n(L)$. Furthermore since L is a symmetric matrix, the sum of its eigenvalues equals its trace, which is equal to the sum of node degrees $n\tilde{d}(\mathcal{G})$. Hence, (9) implies

$$\frac{(n-1)^2}{n\tilde{d}(\mathcal{G})} \leq \sum_{i=2}^{n}\frac{1}{\lambda_i(L)}. \tag{10}$$

Due to (2) and (10),

$$\mathcal{H}^*(\mathcal{G}) = \frac{1}{2n}\sum_{i=2}^{n}\frac{1}{\lambda_i(L)} \geq \frac{(n-1)^2}{2\tilde{d}(\mathcal{G})n^2}. \tag{11}$$

Note that the harmonic mean equals the arithmetic mean if and only if all the numbers are equal. Hence, (9) holds with equality if and only if $\lambda_2(L) = \lambda_3(L) = \ldots = \lambda_n(L)$. Furthermore, all the positive Laplacian eigenvalues of a connected graph are equal if and only if the graph is a complete graph (e.g., see [21]). Hence, (11) holds with equality if and only if $\mathcal{G}$ is a complete graph. Alternatively, the lower bound can also be proved by using (4) and the inequality shown in [22].

(Upper bound:) The Kirchoff index satisfies

$$K_f(\mathcal{G}) \leq \sum_{1\leq i<j\leq n} \delta_{ij}, \tag{12}$$

where the δ_{ij} denotes the distance between nodes i and j, and (12) holds with equality if and only if $\mathcal{G}$ is a tree (e.g., see [14]). Since the sum of distances between the nodes satisfy

$$\sum_{1\leq i<j\leq n} \delta_{ij} = \frac{n(n-1)\tilde{\delta}(\mathcal{G})}{2},$$

(5) and (12) together imply

$$\mathcal{H}^*(\mathcal{G}) \leq \frac{\tilde{\delta}(\mathcal{G})(n-1)}{4n}. \tag{13}$$

Furthermore, since (12) holds with equality if and only if $\mathcal{G}$ is a tree, the same is true for the inequality in (13). $\square$

C. Graphs with Extremal Robustness Scaling

One of the important considerations when designing large scale networks is how the robustness of the system would scale with its size. As indicated by (6)-(7), different network topologies may exhibit different robustness scaling properties. For instance, while the structural vulnerability of complete graph, $\mathcal{H}^*(\mathcal{K}_n)$, tends to zero as the network size increases (see (7)), the structural vulnerability of path graph, $\mathcal{H}^*(\mathcal{P}_n)$, tends to infinity as the network size increases (see (6)). Apart from these two extremal cases of robustness scaling, there are also networks (e.g., star graph) such that $\mathcal{H}^*(\mathcal{G}_n)$ converges to some non-zero value as the network size increases. One question of interest is then which topological properties determine how $\mathcal{H}^*(\mathcal{G}_n)$ behaves as the size increases. In this regard, the following result provides a graph topological characterization of networks with different types of robustness scaling.

Theorem 3.3. *Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ denote an infinite sequence of connected undirected graphs with n nodes. The structural vulnerability of $\mathcal{G}_n$ tends to zero as n goes to infinity only if the average node degree grows unbounded, i.e.,*

$$\lim_{n\to\infty} \mathcal{H}^*(\mathcal{G}_n) = 0 \Rightarrow \lim_{n\to\infty} \tilde{d}(\mathcal{G}_n) = \infty.$$

Furthermore, the structural vulnerability grows unbounded only if the average distance also grows unbounded, i.e.,

$$\lim_{n\to\infty} \mathcal{H}^*(\mathcal{G}_n) = \infty \Rightarrow \lim_{n\to\infty} \tilde{\delta}(\mathcal{G}_n) = \infty.$$

Proof: ($\mathcal{H}^*(\mathcal{G}_n) \to 0$): Note that the lower bound in (8) is non-negative for any connected undirected $\mathcal{G}$ with $n \geq 2$ nodes. Hence, due to the squeeze theorem, if $\mathcal{H}^*(\mathcal{G}_n)$ tends to zero then the lower bound must also tend to zero, i.e.,

$$\lim_{n\to\infty} \mathcal{H}^*(\mathcal{G}_n) = 0 \Rightarrow \lim_{n\to\infty} \frac{(n-1)^2}{2\tilde{d}(\mathcal{G}_n)n^2} = 0. \tag{14}$$

Since the average node degree $\tilde{d}(\mathcal{G}_n)$ is always non-negative, (14) implies

$$\lim_{n\to\infty} \tilde{d}(\mathcal{G}_n) = \infty.$$

($\mathcal{H}^*(\mathcal{G}_n) \to \infty$): If $\mathcal{H}^*(\mathcal{G}_n)$ diverges as n goes to infinity, the upper bound in (8) must also diverge, which is only possible if $\tilde{\delta}(\mathcal{G}_n)$ diverges. $\square$

D. Price of Structural Robustness

In light of the lower bound in Theorem 3.2, the average degree of a network imposes a fundamental limit on how good the structural robustness can be. For instance, as shown in Theorem 3.3, the structural vulnerability to noise can disappear with increasing size only if the average degree grows unbounded. Accordingly, we observe that the graphs with good structural robustness typically pay the price in terms of sparsity, which is also a desirable property in networks since each edge denotes some communications, sensing, or a physical link between the corresponding agents. As such, dense graphs require more resources. That is one of the main reasons why in most cases complete graph is not a feasible network structure although it has the best possible structural robustness as per (7).

In this part, we provide a tight bound that highlights the trade-off between structural robustness and sparsity. To do that, we first consider the best structural robustness that can be obtained with the minimum number of edges. For connected graphs, minimum sparsity is observed in trees, i.e., graphs with n nodes and $n-1$ edges. Note that any graph with fewer edges has to be disconnected. In light of Theorem 3.2, the structural vulnerability of any tree is determined by the average distance between nodes, $\tilde{\delta}$. Furthermore, $\mathcal{H}^*(\mathcal{G})$ is a monotonically increasing function of $\tilde{\delta}(\mathcal{G})$ for trees as per the upper bound in (8). As such, it can be immediately shown that the star graph $\mathcal{S}_n$, has the best structural robustness among the trees as given in (7). Despite being a member of the sparsest family of connected graphs (i.e., trees), the star graph exhibits a very good level of structural robustness. Unlike the path graph, the structural vulnerability of star graph does not grow unbounded as n goes to infinity. Instead it converges to $1/2$. Our next result shows that the average degree of a graph defines a tight upper bound on how smaller its structural vulnerability can be compared to the star graph.

Theorem 3.4. *For any connected undirected graph with n nodes, $\mathcal{G}_n$, and the star graph with n nodes, $\mathcal{S}_n$,*

$$\frac{\mathcal{H}^*(\mathcal{S}_n)}{\mathcal{H}^*(\mathcal{G}_n)} \leq \tilde{d}(\mathcal{G}_n). \tag{15}$$

Furthermore, this bound is tight.

Proof: The bound follows from (7) and the lower bound in (8). The tightness follows from the fact that (15) is satisfied with equality for the complete graph, $\mathcal{G}_n = \mathcal{K}_n$, as per (7) since $\tilde{d}(\mathcal{K}_n) = n-1$. $\square$

Since $\mathcal{S}_n$ has the best structural robustness achievable with the minimum number of edges a connected graph can have, (15) highlights the price of structural robustness in terms of sparsity. Any graph with a significantly better structural robustness (smaller $\mathcal{H}^*(\mathcal{G}_n)$) than the star graph of same size should have a proportionally high average degree.

E. Structural Robustness of Random Regular Graphs

In this subsection, we investigate the structural robustness of random regular graphs and show that they typically have near optimal structural robustness among the graphs of same size and sparsity. A graph is called a k-regular graph if the number of edges incident to each node (the degree) is equal to k. For connected regular graphs with $n \geq 3$ nodes, the feasible values of k are $\{2, 3, \ldots, n-1\}$ with the constraint that n and k can not be both odd numbers since the number of edges is equal to $nk/2$. The complete graph, which has the best structural robustness possible as given in (7), is the k-regular graph with $k = n-1$. We will show that most k-regular graphs have desirable structural robustness properties, except for the special case of $k = 2$, which is the cycle graph $\mathcal{C}_n$. In light of (6), $\mathcal{H}^*(\mathcal{C}_n)$ clearly grows unbounded as the network size increases. Furthermore, the structural vulnerability of a cycle graph is equal to the half of the path graph's, i.e., $\mathcal{H}^*(\mathcal{C}_n) = \mathcal{H}^*(\mathcal{P}_n)/2$. Hence, the structural robustness of a cycle is always within a constant factor of the worst possible among the graphs of equal size.

On the other hand, the structural robustness of k-regular graphs for $k \geq 3$ is significantly different from the cycle graph's structural robustness. As n goes to infinity, for $k \geq 3$ almost every k-regular graph has $\lambda_2(L) \geq k - 2\sqrt{k-1} - \epsilon$ for any $\epsilon > 0$ (e.g., see [23] and the references therein). In light of (4), this property implies an upper bound on the structural vulnerability of those graphs since for any graph

$$\frac{1}{2n}\sum_{i=2}^{n}\frac{1}{\lambda_i(L)} \leq \frac{n-1}{2n\lambda_2(L)}.$$

Accordingly, for any integer $k \geq 3$ and $\epsilon \in (0, k-2\sqrt{k-1})$

$$\lim_{n\to\infty} \Pr\left\{\mathcal{H}^*(\mathcal{G}_{n,k}) \leq \frac{n-1}{2n(k-2\sqrt{k-1}-\epsilon)}\right\} = 1, \quad (16)$$

where $\mathcal{G}_{n,k}$ is a random k-regular graph, i.e., a graph that is selected uniformly at random from the set of all k-regular graphs with n nodes. Since n and k cannot be both odd, for odd values of k the limit in (16) is defined along the sequence of even integers $n \in \{k+1, k+3, \ldots\}$.

By combining (16) with the lower bound in (8) for $\tilde{d}(\mathcal{G}) = k$, we can show that for large values of n, with high probability, the structural vulnerability of random k-regular graphs ($k \geq 3$) is within a constant factor of the smallest possible value among the graphs with the same size and average degree. Furthermore, this factor gets arbitrarily close to one as k increases. In other words, for large values of k, random k-regular graphs have structural robustness arbitrarily close to the best possible (with that many edges) with arbitrarily high probability as the network size increases.

Theorem 3.5. *For any $\epsilon \in (0, k-2\sqrt{k-1})$ and any integer $k \geq 3$*

$$\lim_{n\to\infty} \Pr\left\{\frac{\mathcal{H}^*(\mathcal{G}_{n,k})}{\min\limits_{\mathcal{G}_n:\tilde{d}(\mathcal{G}_n)=k} \mathcal{H}^*(\mathcal{G}_n)} \leq \frac{k}{k-2\sqrt{k-1}-\epsilon} + \epsilon\right\} = 1, \quad (17)$$

where $\mathcal{G}_{n,k}$ is a random k-regular graph.

Proof: Using the lower bound in (8) and (7), for any undirected graph $\mathcal{G}_n$ with n nodes and average degree $\tilde{d}(\mathcal{G}_n) = k$,

$$\min_{\mathcal{G}_n:\tilde{d}(\mathcal{G}_n)=k} \mathcal{H}^*(\mathcal{G}_n) \geq \frac{(n-1)^2}{2kn^2} \quad (18)$$

Using (18) with (16), for any random k-regular graph with $k \geq 3$ and $\epsilon \in (0, k-2\sqrt{k-1})$,

$$\lim_{n\to\infty} \Pr\left\{\frac{\mathcal{H}^*(\mathcal{G}_{n,k})}{\min\limits_{\mathcal{G}_n:\tilde{d}(\mathcal{G}_n)=k} \mathcal{H}^*(\mathcal{G}_n)} \leq \frac{2kn^2}{(2n^2-2n)(k-2\sqrt{k-1}-\epsilon)}\right\} = 1. \quad (19)$$

Note that the upper bound in (19) satisfies

$$\lim_{n\to\infty} \frac{2kn^2}{(2n^2-2n)(k-2\sqrt{k-1}-\epsilon)} = \frac{k}{k-2\sqrt{k-1}-\epsilon}.$$

Due to the definition of limit, there exists some n beyond which the upper bound in (19) is smaller than its limit plus ϵ. Consequently, we can replace the upper bound with that value and obtain (17). $\square$

In light of Theorem 3.5, random k-regular graphs with large k and size have almost optimal structural robustness among the graphs of same size and average degree.

IV. Simulation Results

We simulate the dynamics in (1) for uniform edge weights $w = \mathbf{1}$ on different network topologies to demonstrate their structural robustness. More specifically, we consider the path, complete, and random 3-regular graphs with 20 nodes. In each simulation, the network is initialized at $x(0) = \mathbf{0}$, and the variance of $x(t)$ is observed under the noisy consensus dynamics as per (1), where $\xi(t) \in \mathbb{R}^n$ is white Gaussian noise with zero mean and unit covariance. The results are shown in Fig. 1. The average of state variances over the whole horizon were observed as 1.64 ($\mathcal{P}_{20}$), 0.29 ($\mathcal{G}_{20,3}$), and 0.03 ($\mathcal{K}_{20}$). These values are aligned with the theoretical results. The average distance on a path graph is known to satisfy $\tilde{\delta}(\mathcal{P}_n) = (n+1)/3$. Accordingly, the upper bound in (8), which should satisfy with equality for the path graph, can be computed as 1.66. Similarly, computing the lower bound for $n = 20$ and $\tilde{d}(\mathcal{K}_{20}) = 19$ results in 0.024, which is close to the observed value in the simulation. For the random 3-regular graph used in the simulation, the average distance

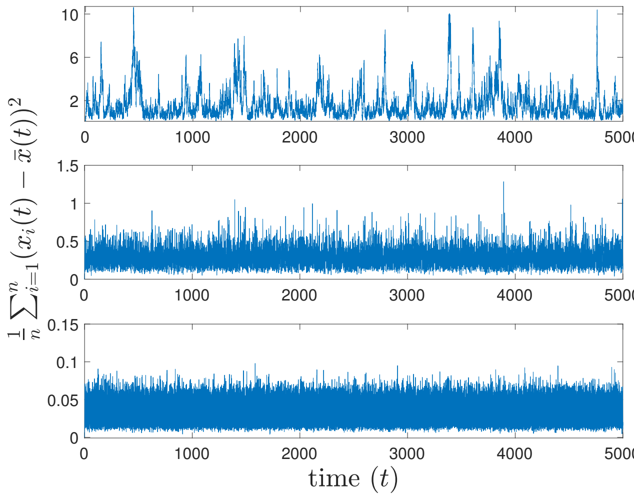

Fig. 1. Variance of states under the noisy consensus dynamics on path (top), random 3-regular (middle), and complete (bottom). Graphs have 20 nodes and the edge weights are all set to one to illustrate structural robustness.

was computed as 2.56. Accordingly, the lower bound and the upper bound in (8) were computed as 0.15 and 0.6.

V. Conclusion

We investigated the structural robustness of undirected linear consensus networks to noisy interactions. We measured the structural robustness of a graph based on the smallest possible value of the expected steady state population variance of states under the noisy consensus dynamics with admissible edge weights from the unit interval. We showed that the average distance and the average node degree in the underlying graph define tight bounds on the structural robustness. Using these bounds, we also characterized the graphs with extremal robustness scaling. We then presented a fundamental trade-off between the structural robustness and the average degree of networks. We expressed this trade-off in terms of a tight bound on the ratio of structural robustness of any given graph to the structural robustness of the star graph of same size, which has the best structural robustness among the connected graphs with minimum average degree (trees). We also showed that random k-regular graphs with n nodes typically have near-optimal structural robustness among the graphs of size n and average degree k.

As a future direction, we intend to extend our robustness analysis to the generalized case of directed graphs, where the interactions between nodes are not necessarily symmetric. We also plan to investigate the fundamental relationships between the structural robustness and other system properties. For example, recently it was shown that the distances between nodes have a major impact on the controllability of consensus networks and there are trade-offs between the controllability and robustness of such systems (e.g., [24], [25]). We believe that the results in this paper can be used for further investigation of such relationships.

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