Toronto Math Circles: Junior
Third Annual Christmas Mathematics Competition
Solutions

1. A four digit number A, when read from left to right, consists of 4 consecutive increasing integers. Determine the sum of the possible values of A.
   Solution. $\boxed{24069}$
   Note that the thousand's digit can be $1, 2, \ldots, 6$. Each of these digits implies that the hundred's digit can be $2, 3, \ldots, 7$. Each of these digits implies that the ten's digit can be $3, 4, \ldots, 8$. Each of these digits implies that the unit's digit can be $4, 5, \ldots, 9$. Therefore, the sum of these values is

$$\begin{aligned}
&1000(1+2+\cdots+6)+100(2+3+\cdots+7)+10(3+4+\cdots+8)+(4+5+\cdots+9)\\
=&1000\frac{6(7)}{2}+100\frac{6(9)}{2}+10\frac{6(11)}{2}+\frac{6(13)}{2}\\
=&24069
\end{aligned}$$

2. Determine if there exists a 3 digit number n, such that $600 \le n \le 699$ and removing the ten's digit yields a two digit number that is 6 less than $\frac{1}{6}$ of the original number.
   Solution 1. $\boxed{\text{No such number exists}}$
   Let the three numbers be $600 + 10A + B$ where $0 \le A, B \le 9$ then

$$\begin{aligned}
\frac{1}{6}(600+10A+B)-6&=60+B\\
600+10A+B&=396+6B\\
204+10A&=5B\\
204&=5(B-2A)
\end{aligned}$$

   Observe that the right hand side is divisible by 5 but the left hand side is not. Therefore, no such number exists.
   Solution 2.
   The smallest such number is 600 and $\frac{1}{6}(600) - 6 = 94 > 69$. Therefore, no such number exists.

3. Two rectangles are drawn on the plane such that one rectangle is entirely contained in the other. Describe a straight edge construction of a line that equally divides the area of the region between the two rectangles. Be sure to explain why your construction works.
   Solution. Let the four corners of one rectangle be labeled as A, B, C, D and the four corners of the other rectangle be labeled as E, F, G, H. Use the straight edge to draw the segments AC, BD, EG, FH. Label the intersection of AC and BD as X and label the intersection of EG and FH as Y. Use the straight edge to draw a line that passes through XY. This line equally divides the area of the region between the rectangles.
   To see that this line works, first recall that the intersection of the diagonals of a rectangle is the center of the rectangle. Thus, X and Y are the center of $ABCD$ and $EFGH$, respectively. Next, recall that any line that pass through the center of a rectangle equally divides the area of the rectangle. Thus, the line through X and Y will divide $ABCD$ into two regions P and Q and divide $EFGH$ into two regions R and S such that $P = Q$ and $R = S$. Therefore, $P - R = Q - S$ is the region between the two rectangles.

4. A sequence of number is called **COOL** if every term starting from the third term is the average of all the previous terms. A particular COOL sequence has the first term as 1 and the 2017^{th} term as 2017. Determine the value of the second term.
   Solution. $\boxed{4033}$ Let the second term be x and let this sequence be a_n. Proceed by induction to show that the n^{th} term of this COOL sequence is always $\frac{x+1}{2}$.
   Base Case: $n = 3$. By definition of a COOL sequence, the third term is $a_3 = \frac{x+1}{2}$.
   Induction Hypothesis: Assume the result is true for $n = k$.

Induction Step: Let the sum of the first k terms be S_k. Then let then the value of the $k+1$ terms is

$$a_{k+1} = \frac{S_k}{k} = \frac{S_{k-1} + a_k}{k} = \frac{a_k(k-1) + a_k}{k} = \frac{ka_k}{k} = a_k = \frac{x+1}{2}$$

Therefore, by induction, $a_{2017} = 2017 = \frac{x+1}{2}$. Solving for x yields 4033 as the second term.

5. On the Cartesian plane, each lattice point is assigned an integer such that, for any square with its sides parallel to two axis, the sum of values at the vertices is equal to the number of lattice points in this square. For example, $(-1,1),(1,1),(1,-1),(-1,-1)$ has 9 lattice points inside the square. Prove that no such assignment exists. Note: A lattice point is a point (x,y) such that x and y are both integers. *Solution.* For the sake of contradictions, assume that such an assignment exists. Label the following 9 lattice points,

$$A(-1,1), B(0,1), C(1,1), D(-1,0), E(0,0), F(1,0), G(-1,1), H(0,0).I(1,-1)$$

Observe that

$$\begin{aligned} A+B+D+E &= 4 \\ B+E+H+G &= 4 \\ B+C+F+E &= 4 \\ E+F+I+H &= 4 \end{aligned}$$

Adding theses four equations yields

$$(A+C+G+I) + 2\,(B+F+H+D) + 4E = 16$$

Since $A+C+G+I = 9$ then

$$2\,(B+F+H+D) + 4E = 7$$

This is a contradiction because the left hand side is even while the right hand side is odd. Therefore, no such assignment exists.

Toronto Math Circles: Senior
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Solutions

1. Let a and b be two numbers such that $b = a+2$. Consider two monic quadratics such that their vertices are at $A(a, 0)$ and $B(b, 0)$. Given that these two quadratics only intersect at one point, C. Determine the area of the triangle ABC.
   Note: A quadratic is monic if the coefficient of x^2 is 1.
   Solution. $\boxed{1}$ Since the vertices are on the x-axis, the two quadratics can be written as

   $$\begin{cases} y = (x-a)^2 \\ y = (x-b)^2 \end{cases}$$

   Equating the two equations, expanding and simplifying yields

   $$-2ax + a^2 = -2bx + b^2$$

   Solving for x yields

   $$x = \frac{1}{2}\frac{b^2-a^2}{b-a} = \frac{1}{2}(a+b)$$

   Substituting this back into either of the original equation yields

   $$y = \frac{1}{4}(b-a)^2$$

   Therefore, the area of the triangle is

   $$\frac{1}{2}(b-a)\left(\frac{1}{4}(b-a)^2\right) = \frac{(b-a)^3}{8} = \frac{2^3}{8} = 1$$

   Remark. It suffices by assuming $a = 0$ because the two quadratics and the triangle will remain unchanged from a translation.

2. On the Cartesian plane, each lattice point is assigned an integer value such that, for any square with positive area and its four vertices on lattice points, the sum of values at the vertices is equal to the area of the square. Prove that no such assignment exists. Note: A lattice point is a point (x, y) such that x and y are both integers.
   Solution 1. For the sake of contradictions, assume that such an assignment exists. Let n be a positive integer and label the following 9 lattice points,

   $$A(-n,n), B(0,n), C(n,n), D(-n,0), E(0,0), F(n,0), G(-n,-n), H(0,-n).I(n,-n)$$

   Observe that

   $$\begin{aligned} A+B+D+E &= n^2 \\ B+E+H+G &= n^2 \\ B+C+F+E &= n^2 \\ E+F+I+H &= n^2 \end{aligned}$$

   Adding theses four equations yields

   $$(A+C+G+I) + 2(B+F+H+D) + 4E = 4n^2$$

   Since $A+C+G+I = (2n)^2$ and $B+F+H+D = 2n^2$

   $$2n^2 + 2E = 0$$

Thus, $E = -n^2$ for every positive integer n, which is not possible. Therefore, no such assignment exists.
Solution 2. For the sake of contradictions, assume that such an assignment exists. Label the following 9 lattice points,

$$A(-1,1), B(0,1), C(1,1), D(-1,0), E(0,0), F(1,0), G(-1,-1), H(0,-1).I(1,-1)$$

Observe that

$$\begin{aligned} A+B+D+E &= 1 \\ B+E+H+G &= 1 \\ B+C+F+E &= 1 \\ E+F+I+H &= 1 \end{aligned}$$

Adding theses four equations yields

$$(A+C+G+I)+2(B+F+H+D)+4E=4$$

Since $A+C+G+I=4$ and $B+F+H+D=2$

$$2+2E=0$$

Thus, $E=-1$. By shifting the center of the square to any other lattice point, it can be shown that the value at every point is -1, which is a contradiction. Therefore, no such assignment exists.
Remark. The question can strengthened by requiring all sides of the square to be parallel to the x and y axises except the proofs above will not be the same.

3. Let n be an odd positive integer not divisible by 5. Prove that in the following n numbers

$$1, 11, \ldots, \underbrace{11\cdots11}_{n}$$

there is one that is divisible by n.
Solution. Assume that none of the n numbers are divisible by n then in modulo n, then each number must be congruent to $1, 2, 3, \ldots, n-1$. By pigeonhole principal, there exists two numbers which has the same remainder. Let these two numbers be $i < j$ By taking their difference,

$$\underbrace{11\cdots11}_{j} - \underbrace{11\cdots11}_{i} = \underbrace{11\cdots11}_{j-i} \times 1\underbrace{0\ldots0}_{i}$$

must be divisible by n. Since n is odd and not divisible by 5,

$$\gcd\left(n, 1\underbrace{0\ldots0}_{i}\right) = 1$$

Therefore, n must divide $\underbrace{11\cdots11}_{j-i}$, which is a contradiction. Therefore, one of the n original numbers must be divisible by n.

4. Let AB be the diameter of a semi-circle with center O. Point C is the midpoint of arc AB and M is the midpoint of the chord AC. If CH is perpendicular to BM at H, prove that $CH^2 = AH \cdot OH$. Be sure to include a clearly labeled diagram.
Solution. First connect OC and BC then $\angle CHB = \angle COB = 90°$, which implies that $CHOB$ is a cyclic quadrilateral. Therefore,

$$\angle OHB = \angle OCB = 45°$$

Since $\triangle CMB \sim \triangle HMC$ then

$$AM^2 = CM^2 = MH \cdot MB$$

This implies that $\triangle AMH \sim \triangle BMA$ then

$$\angle MAH = \angle MBA, \angle AHM = \angle BAM = 45^\circ$$

Therefore, $\triangle AMH \sim \triangle BOH$. This implies that

$$AH \cdot OH = MH \cdot BH$$

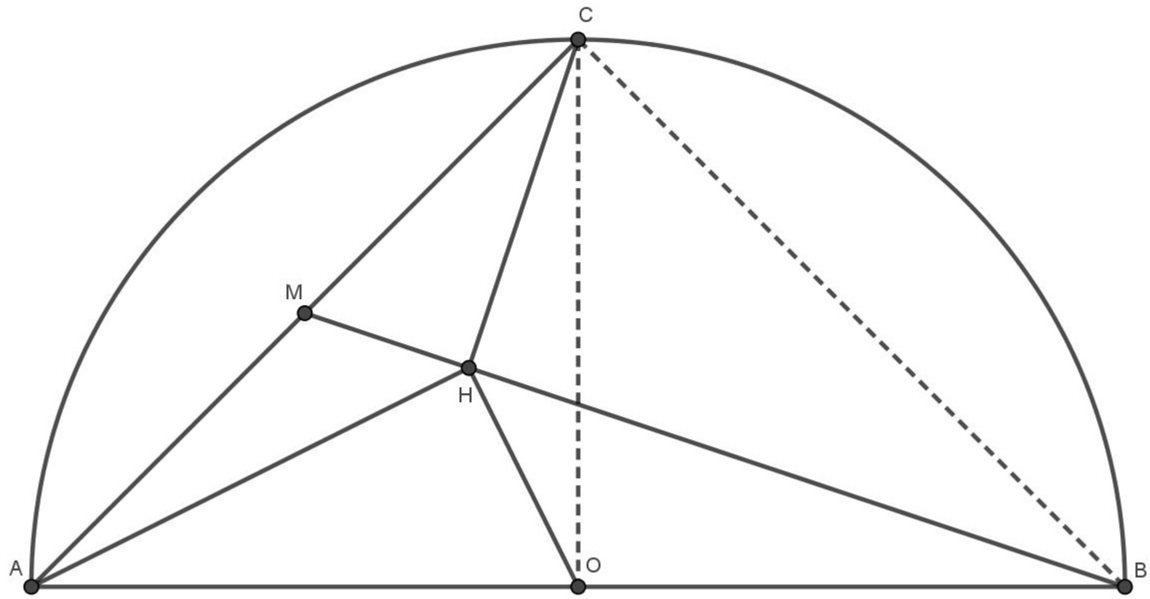
Since $\triangle CMH \sim \triangle BCH$ then

$$CH^2 = MH \cdot BH$$

Finally, putting everything together,

$$CH^2 = MH \cdot BH = AH \cdot OH$$

Remark. The problem can also be solved by using analytic geometry.



5. Let c and d be integers with $c \neq 0$. Prove that c divides d if and only if there are infinitely many distinct integral pairs (x, y) such that x and y both divide $c(x+y)+d$.
Solution. Suppose that $c|d$ then there exists an integer k such that $d = ck$. Define the infinitely many distinct integral pairs as

$$(x_n, y_n) = (n, -n-k)$$

for $n = 1, 2, \ldots$. It's easy to see that every such pair satisfy the desired result.
Conversely, let (x_n, y_n) be a sequence of infinitely many distinct integral pairs such that x_n and y_n both divide $c\,(x_n + y_n) + d$. Note that this is equivalent to $x_n|\,(cy_n + d)$ and $y_n|\,(cx_n + d)$. If $|c| = 1$ or $d = 0$ then the result is obviously true. Therefore, let $d \neq 0$ and $|c| \geq 2$.
Observe that there must exist some positive integer k such that $|x_k| > c^4d^4$ or $|y_k| > c^4d^4$. Without loss of generality, assume that $|x_k| > c^4d^4$. Since $x_k|\,(cy_k + d)$ then

$$|cy_k + d| \geq |x_k| > c^4d^4$$

By triangle inequality, $|cy_k| > c^4d^4 - |d|$. Since $|c| \geq 2$ then

$$|y_k| > \frac{1}{|c|}\left(c^4d^4 - |d|\right) \geq \frac{1}{|c|}\left(2|c|^3d^4 - |d|\right) = 2c^2d^4 - \frac{|d|}{|c|}$$

Since $c^2d^4 - \frac{|d|}{|c|} \geq 0$ then

$$|y_k| > c^2d^4$$

Since $x_k | (cy_k + d)$ and $y_k | (cx_k + d)$ then

$$\frac{(cy_k + d)(cx_k + d)}{x_k y_k} = c^2 + \frac{cd}{x_k} + \frac{cd}{y_k} + \frac{d^2}{x_k y_k} \in \mathbb{Z}$$

Since $c^2 \in \mathbb{Z}$ then $\frac{cd}{x_k} + \frac{cd}{y_k} + \frac{d^2}{x_k y_k} \in \mathbb{Z}$. Combining this with the following:

$$\begin{aligned}
\left| \frac{cd}{x_k} + \frac{cd}{y_k} + \frac{d^2}{x_k y_k} \right| &\leq \left| \frac{cd}{x_k} \right| + \left| \frac{cd}{y_k} \right| + \left| \frac{d^2}{x_k} \right| \\
&< \frac{|cd|}{c^4 d^4} + \frac{|cd|}{c^2 d^4} + \frac{d^2}{c^4 d^4} \\
&= \frac{1}{|c|^3 |d|^3} + \frac{1}{|c||d|^3} + \frac{1}{c^4 d^2} \\
&\leq \frac{1}{|c|^3} + \frac{1}{|c|} + \frac{1}{c^6} \\
&\leq \frac{1}{2^3} + \frac{1}{2} + \frac{1}{2^6} \\
&< 1
\end{aligned}$$

shows that $\frac{cd}{x_k} + \frac{cd}{y_k} + \frac{d^2}{x_k y_k} = 0$. Therefore,

$$\frac{(cy_k + d)(cx_k + d)}{x_k y_k} = c^2$$

For the sake of contradiction, assume that $c \nmid d$ then there exists a prime p and a positive integer a such that $p^a | c$ but $p^a \nmid d$. Since $p^{2a} | c^2$ then by pigeonhole principle, one of $\frac{cy_k + d}{x_k}$ or $\frac{cx_k + d}{y_k}$ is divisible by p^a. Without loss of generality, assume that $p^a x_k | (cy_k + d)$ then $p^a | (cy_k + d)$. This is a contradiction to $p^a \nmid d$. Therefore, $c | d$.