Goursat problem for some hyperbolic equations solved by Laplace transform method

Mircea Ion CÎRNU
Department of Applied Mathematics
Faculty of Applied Sciences
Polytechnic University, Bucharest, Romania

Abstract
This paper presents the solution of a Goursat problem for homogeneous hyperbolic partial differential equations by using the Laplace transform method. The solutions are obtained as finite sums of elementary functions. Beside the scientific purpose, the paper also has a didactical one, since the use of the Laplace transform to solve partial differential equation generally presents great difficulty to students. There are very few exercises of this type, which are generally difficult to solve. However, the class of hyperbolic equations solved here by the Laplace transform method contains a wide variety of particular cases that can be easily solved by the aforementioned method. Due to the fact that the considered equations are complete solvable in elementary functions, the students do not require additional knowledge. The paper continues the Author’s concerns about the Laplace transform, recorded in the Introduction.

Keywords: partial differential equations; homogeneous linear hyperbolic equations; Goursat problem; Laplace transform.

2010 Mathematics Subject Classification: 35A22. ZDM Subject Classification: I70.

1. INTRODUCTION:

The Laplace transform is one of the most powerful methods for solving different types of equations. Initially considered by Leonard Euler in 1737, it was extensively used by Pierre-Simon Laplace in his work on probability theory. Having applications in various fields of science, its importance is increasing rapidly.

In the following, we shortly mention the Author’s contributions to the study and to applications of the Laplace transform. In the series of papers [4], the Laplace transform was extended to distributional functions, namely to functions having distributions (generalized functions) as values. In [5], it was given a closed formula for the numerator of the Laplace transform of the solution of the initial value problem for a homogeneous linear differential equation with constant coefficients, its denominator being, as it is known, the characteristic polynomial of the equation. The famous Girard-Newton recurrence formulas about the sums of power of roots of a polynomial, were shortly deduced in that paper, using the above-mentioned formula. This gave a new example, in addition to calculating the matrix exponential or to the results in [13], for obtaining purely algebraic results using the Laplace transform. In turn, the Girard-Newton formulas were used in [9] to the deduction of automatic formulas, based on discrete deconvolution, [12], for numerical calculus of such sums and these were used also in [9] to obtain a new method for the approximate calculus of the roots of a polynomial, similar to the method of Daniel Bernoulli [18], but much easier because it has removed the use of the difference equation on which Bernoulli’s method is based on. Following an idea of J. H. He (see [16], [17] and [22]), in paper [10], a generalization of the Newton-Raphson method for solving transcendental algebraic equations was given, by their reduction to a “resolving” polynomial equation which gives the variations of the iterations of the exact solution. As one can see in [19], this method is more powerful than the Newton-Raphson method, being deduced by Taylor development of high order. All results mentioned here, except the first one, were included in the recent Author’s books [2] and [3].
Another topic recently taken into consideration is the hybrid Laplace transformation, defined by Valeriu Prepelita in [20] by composing the usual Laplace and Z transformations. It applies to the functions that have both continuous and discrete variables, thus increasing the possibilities of applications of the transform. In the recent paper [11], written in collaboration with V. Prepelita, the main properties of the hybrid transformation are demonstrated, using the corresponding properties of the usual Laplace and Z transformations. In the same paper, it is exemplified the increased power of the hybrid transformation, by quickly solving a recurrent integral equation with discrete auto-convolution, previously solved by the present Author in [8], reducing it by the usual Laplace transform to a recurrence differential equation with discrete auto-convolution, which in turn was solved by the methods given in his earlier works [7] and [6].

2. MATERIALS AND METHODS:

In the paper [21], E. J. Scott, considered the Goursat problem composed by the homogeneous linear hyperbolic partial differential equation, with the coefficients depending on the same variable, say \( y \),

\[
\frac{\partial^2 v(x, y)}{\partial x \partial y} + a(y) \frac{\partial v(x, y)}{\partial x} + b(y) \frac{\partial v(x, y)}{\partial y} + c(y) v(x, y) = 0, \quad x \geq 0, \tag{1}
\]

and the conditions

\[
\frac{dv(x,0)}{dx} + p v(x,0) = 0, \quad \frac{dv(0,y)}{dy} + a(y) v(0,y) = 0, \quad v(0,0) = v_0, \tag{2, a-c}
\]

where \( a(y), b(y) \) and \( c(y) \) are given functions, \( a(y) \) integrable with primitive \( A(y) = \int_0^y a(t) dt, \)

\( p \) and \( v_0 \) are complex numbers, the unknown \( v(x, y) \) being Laplace transformable in the variable \( x \).

Using the partial Laplace transform \( L_x (v(x, y)) = \int_0^\infty e^{-zx} v(x, y) dx = \hat{v}(z, y), \) E. J. Scott showed that in case \( c(y) = a(y) b(y) + b'(y), \) the problem (1), (2) can be solved in elementary functions, while if the coefficients \( a, b, c \) are constants and \( c \neq ab \), the solution contains the Bessel function \( J_n \).

We extend here the first Scott’s result showing that if \( c(y) = a(y) b(y) - nb'(y), \)

\[
c(y) = a(y) b(y) - nb'(y), \tag{3}
\]

where \( n \) is an arbitrary fixed integer, the problem (1), (2) also can be solved by Laplace transform in finite sums of elementary functions.

We also give several examples of Goursat problems of the type considered here and their results, which are easy to solve by Laplace transform method.

The Goursat problems can be used to Riemann function method for solving non-homogeneous hyperbolic equations. See for example [15].

We will use in the next Section the following Laplace transform formulas ([14])

\[
L_x \left( \frac{\partial v(x, y)}{\partial x} \right) = z \hat{v}(z, y) - v(0, y), \quad L_x \left( e^{-ax} v(x) \right) = \hat{v}(z + a), \quad L_x \left( \frac{x^n}{n!} \right) = \frac{1}{z^{n+1}},
\]

hence \( L_x \left( \frac{e^{-ax} x^n}{n!} \right) = \frac{1}{(z + a)^{n+1}}, \) where \( n \) is a natural and \( a \) a complex number.

3. RESULTS AND DISCUSSION:

In this Section we prove the following
Theorem. The Goursat problem (1), (2), with the coefficient \(c(y)\) given by (3), has the solution:

1) \(v(x, y) = v_0 e^{-\mu x (y)}\), for \(n = 0\), \hspace{1cm} (4)

2) \(v(x, y) = v_0 e^{-b(0)x (y)} \sum_{k=0}^{n} \binom{n}{k} (b(y) - b(0))^k x^k k! = 0\), for \(n > 0\), \(p = b(0)\), \hspace{1cm} (5)

3) \(v(x, y) = v_0 e^{-\alpha (y)} \frac{(b(y) - p)^n}{(b(0) - p)^n} \left[ e^{-px} - \sum_{k=0}^{n} \left( \frac{b(0) - p}{b(y) - p} \right)^k \frac{k!}{j!} \right] e^{-\alpha y} \), for \(n > 0\), \(p \neq b(0)\), \hspace{1cm} (6)

4) \(v(x, y) = v_0 e^{-b(0)} \sum_{k=0}^{n-1} \binom{m-1}{k} (b(0) - b(y))^k x^k k! = 0\), for \(n = -m < 0\), \(p = b(0)\), \hspace{1cm} (7)

5) \(v(x, y) = v_0 e^{-\alpha (y)} \frac{(b(0) - p)^m}{(b(y) - p)^m} \left[ e^{-px} - \sum_{k=0}^{m} \left( \frac{b(0) - p}{b(y) - p} \right)^k \frac{k!}{j!} \right] e^{-\alpha y} \), for \(n = -m < 0\), \(p \neq b(0)\). \hspace{1cm} (8)

Proof. Applying the Laplace transform \(L_z\) to the equation (1) and using the above mentioned properties of the transform, we get the first order differential equation

\[
\frac{d}{dy} [z \hat{v}(z, y) - v(0, y)] + a(y) [z \hat{v}(z, y) - v(0, y)] + b(y) \frac{d\hat{v}(z, y)}{dy} +
\]

\[+ [a(y)b(y) - nb'(y)] \hat{v}(z, y) = 0.\]

In view of the condition (2, b), the equation becomes

\[(z + b(y)) \frac{d\hat{v}(z, y)}{dy} = -[za(y) + a(y)b(y) - nb'(y)] \hat{v}(z, y),\]

from which, denoting \(q = b(0)\), it results

\[\hat{v}(z, y) = C(z) e^{-\alpha (y)} \left( \frac{z + b(y)}{z + q} \right)^n,\]

where \(C(z)\) is an arbitrary function that will be determined below.

Applying the Laplace transform \(L_z\) to the boundary condition (2,a), it results

\[z \hat{v}(z, 0) - v_0 + p \hat{v}(z, 0) = 0. \hspace{1cm} (11)\]

From relations (11) and (10), the last considered for \(y = 0\), it follows
$C(z) = \hat{v}(z,0) = \frac{V_0}{(z+p)}$, \hspace{1cm} (12)

hence, using (10) and (12), it results that the solution of the differential equation (9) is

$$\hat{v}(z, y) = v_0 e^{-\lambda(y)} \frac{(z + b(y))^n}{(z + p)(z + q)^n}.$$ \hspace{1cm} (13)

We have following cases:

1) If $n = 0$, the relation (13) takes the form $\hat{v}(z, y) = e^{-\lambda(y)} \frac{v_0}{z + p}$, hence using the inverse transform, we obtain the solution (4).

2) For $p = q = b(0)$, using Newton’s binomial formula, the relation (13) becomes

$$\hat{v}(z, y) = v_0 e^{-\lambda(y)} \frac{(z + b(y))^n}{(z + p)^{n+1}} = v_0 e^{-\lambda(y)} \frac{(z + p + b(y) - p)^n}{(z + p)^{n+1}} =$$

$$= v_0 e^{-\lambda(y)} \sum_{k=0}^{n} \binom{n}{k} (b(y) - p)^k (z + p)^{n-k} = v_0 e^{-\lambda(y)} \sum_{k=0}^{n} \binom{n}{k} (b(y) - p)^k (z + p)^{n-k+1}.$$ \hspace{1cm} (14)

Applying the inverse transform it results solution (5).

**Example.** For $a(y) = 1$, $b(y) = -y$, $n = 1$, $p = b(0) = 0$, $v_0 = 1$, from (5) we obtain the solution $v = e^{-\lambda} (1 - xy)$. This case was given in [21].

3) When $p \neq q = b(0)$, we make in (13) the partial fractions decomposition

$$\frac{(z + b(y))^n}{(z + p)(z + q)^n} = \frac{B_0}{z + p} + \sum_{k=1}^{n} \frac{B_k}{(z + q)^k},$$ \hspace{1cm} (14)

where $B_0$, $B_1$, ..., $B_n$ are complex numbers that will be determined below.

Multiplying the relation (14) with $z + p$, is obtained for $z \to -p$,

$$B_0 = \frac{(b(y) - p)^n}{(q - p)^n}.$$ \hspace{1cm} (15)

From (14) and (15) it results, factoring the difference of powers and using again the Newton’s binomial formula,

$$\sum_{k=1}^{n} \frac{B_k}{(z + q)^k} = \frac{1}{z + p} \left[ \frac{(z + b(y))^n}{(z + q)^n} - \frac{(b(y) - p)^n}{(q - p)^n} \right] =$$

$$= \frac{1}{z + p} \left[ \frac{z + b(y) - b(y) - p}{z + q} \right] \sum_{k=1}^{n} \frac{(z + b(y))^{k-1}}{(z + q)^{k-1}} \frac{(b(y) - p)^{n-k}}{(q - p)^{n-k}}.$$
\[(q - b(y))\sum_{k=1}^{n} \frac{(z + q + b(y) - q)^{k-1}(b(y) - p)^{-k}}{(z + q)^{k}} = \]
\[(q - b(y))\sum_{k=1}^{n} \frac{(b(y) - p)^{k}}{(z + q)^{k+1}} \sum_{j=0}^{k-1} \frac{(k - 1) j}{(z + q)^{j}}(b(y) - q)^{j} = \]
\[- \frac{(b(y) - p)^{k}}{(q - p)^{k}} \sum_{k=1}^{n} \frac{(q - p)^{k}}{(b(y) - p)^{k}} \sum_{j=0}^{k-1} \frac{(k - 1) j}{(z + q)^{j}}(b(y) - q)^{j+1}. \quad (16)\]

From (13), (14), (15) and (16), it results
\[\hat{v}(z, y) = v_0 e^{-\lambda(y)} \frac{(b(y) - p)^{m} y}{(z + p)^{m}} \left[ \frac{1}{(z + q)^{m}} - \sum_{k=1}^{n} \frac{(q - p)^{k-1}}{(b(y) - p)^{k}} \sum_{j=0}^{k-1} \frac{(k - 1) j}{(z + q)^{j}}(b(y) - q)^{j+1} \right].\]

Applying the inverse transform, and replacing \( q = b(0) \), we obtain the solution (6).

4) If \( n = -m < 0 \) and \( p = q = b(0) \), the relation (13) takes the form
\[\hat{v}(z, y) = v_0 e^{-\lambda(y)} \frac{(z + p)^{m-1}}{(z + b(y))^{m}} = v_0 e^{-\lambda(y)} \frac{(z + b(y) + p - b(y))^{m-1}}{(z + b(y))^{m}} = \]
\[v_0 e^{-\lambda(y)} \sum_{k=0}^{m-1} \frac{m-1}{k} (p - b(y))^k (z + b(y))^{k+1},\]

hence applying the inverse transform, we obtain the solution (7).

**Example.** If \( \alpha(y) = -\alpha(y) \), \( b(y) = -\beta(y) \), \( n = -1 \), \( p = -\beta(0) = b(0) \), \( v_0 = 1 \), from (15) we obtain the solution \( v = e^{\alpha y} \exp \left[ \int_0^y \alpha(t) dt \right] \). As mentioned in Section 2, this case was given in [21].

5) When \( n = -m < 0 \) and \( p \neq q = b(0) \), the relation (13) takes the form
\[\hat{v}(z, y) = v_0 e^{-\lambda(y)} \frac{(z + q)^m}{(z + p)(z + b(y))^m},\]

that is again (13), with \( m \) instead \( n \) and \( q = b(0) \) inverted with \( b(y) \). Performing this changes, from (6) we obtain (8).

4. CONCLUSION:
The Goursat problem, solved by Laplace transform or other methods, is related to Riemann function method for solving non-homogeneous hyperbolic partial differential equations and therefore to study phenomena governed by such equations. This is the main goal of this paper. But the work has also a didactic purpose. While in the case of applying the Laplace transform method to solve differential equations there are numerous examples of working with the students, at partial differential equations such examples are limited and generally laborious. The present article attempts to improve this situation, solving by Laplace transform method a partial differential equation that has many particular cases that can be easily solved by the proposed method, the solutions being obtained as finite sums of elementary functions. A few such particular cases will be given in the following. We will give only
the problem and the solution resulting from the above theorem, the students being encouraged to solve such problems by Laplace transform method, in a manner similar to the proof of the theorem.

**Particular Cases**

4.1. For the problem

\[
\frac{\partial^2 v(x,y)}{\partial x \partial y} + \alpha \frac{\partial v(x,y)}{\partial x} + (y + \beta) \frac{\partial v(x,y)}{\partial y} + (\alpha y + \gamma) v(x,y) = 0, \quad x \geq 0,
\]

\[
\frac{dv(x,0)}{dx} + p v(x,0) = 0, \quad \frac{dv(0,y)}{dy} + \alpha v(0,y) = 0, \quad \text{where } \alpha, \beta, \gamma \text{ are integers, we have } b(0) = \beta,
\]

the relation (3) being satisfied if \( n = \alpha \beta - \gamma \) and place the solutions mentioned in theorem with necessary customizations. For example, for \( \alpha = 1, \beta = 2 \) and \( \gamma = 1 \), it results \( n = 1 \). In this case, for \( p = 2 \) the problem has the solution \( v(x,y) = v_0 (1 + xy) e^{-2y-x} \), and for \( p \neq 2 \), the solution \( v(x,y) = \left[ (y + 2 - p) e^{-px} - ye^{-2x} \right] e^{-y}, \) with \( v_0 = 2 - p \).

4.2. The Goursat problem

\[
\frac{\partial^2 v(x,y)}{\partial x \partial y} + \frac{1}{y + \alpha} \frac{\partial v(x,y)}{\partial x} + (y + \alpha) \frac{\partial v(x,y)}{\partial y} + 2 v(x,y) = 0, \quad x, y \geq 0,
\]

\[
\frac{dv(x,0)}{dx} + p v(x,0) = 0, \quad \frac{dv(0,y)}{dy} + \frac{1}{y + \alpha} v(0,y) = 0, \quad \text{where } \alpha \neq 0 \text{ is a natural number, has the solution}
\]

\[
v(x,y) = \frac{1}{y + \alpha} e^{-(y+\alpha)}, \quad \text{for } p = \alpha \text{ and } v(x,y) = \frac{(\alpha - p) e^{-px} + ye^{-(y+\alpha)} + (\alpha - p) e^{-x(y+\alpha)}}{(y + \alpha)(y + \alpha - p)}, \quad \text{for } p \neq \alpha.
\]

4.3. The Goursat problem

\[
\frac{\partial^2 v(x,y)}{\partial x \partial y} + \frac{1}{y + \alpha} \frac{\partial v(x,y)}{\partial x} + e^y \frac{\partial v(x,y)}{\partial y} + (\alpha - 1) e^y v(x,y) = 0, \quad x \geq 0,
\]

\[
\frac{dv(x,0)}{dx} + p v(x,0) = 0, \quad \frac{dv(0,y)}{dy} + \alpha v(0,y) = 0, \quad \text{where } \alpha \text{ is an integer, has the solution}
\]

\[
v(x,y) = \left[ 1 + x(e^y - 1) e^{-\alpha y} \right] e^{-px}, \text{ if } p = 1 \text{ and}
\]

\[
v(x,y) = \left[ e^{-px} - (e^y - 1) e^{-x} \right] e^{-\alpha y}, \text{ if } p \neq 1.
\]

5. ACKNOWLEDGEMENTS:

I thank to my student and collaborator Anda Elena Olteanu for the help in drafting the final form of some of my lately work.

6. REFERENCES:


AUTHOR’S BRIEF BIOGRAPHY:

Prof. Mircea Ion CÎRNU: Licensed in 1964, PhD in 1976, professor (retired) at the Polytechnic University of Bucharest, Romania. Published 16 books, 75 articles and held 22 communications. Member of American and Roumanian Mathematical Societies, ISSAC and Editorial board of several Journals. Reviewer to Mathematical Reviews and Zentralblatt für Mathematik. His biography is presented in Marquis Who’s Who in the World. Selected as a Leading Educator of the World.