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Research Article

On the Baire's Category Theorem as an Important Tool in General Topology and Functional Analysis

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Abstract

The problems connected with Topological spaces, Metric spaces, Banach spaces, Boundedness and continuous mappings lie at the heart of Baire's Category theorem. In this paper, we give a short survey on some classical results due to the Baire's Category theorem in the field of general Topology and Functional analysis.

Keywords: Category; Interior of asset; empty interior; Baire space; open mapping; locally compact Hausdorff space; Metric space.

Introduction

The Baire's category theorem is an important tool in general topology and functional analysis. The theorem was proved by Rene'-Louis Baire in his 1899 doctoral thesis [1]. The proof of Baire-category theorem made it possible to derive from it the uniform boundedness theorem as well as the open mapping theorem. The uniform boundedness theorem (or uniform boundedness principle) by S. Banach and H. Steinhaus (1927) [2] is of great importance. Infact throughout analysis there are many instances of results related to this theorem, the earliest being an investigation by H.Lebesgue (1909) [3]. Baire category theorem has various other applications in functional analysis and is the main reason why category enters into numerous proofs for instance the more advace books by R.E Edwards (1965) and J.L Kelley and I. Namiaka (1963) [3,4].

Research Methodology

Definition 1.0 (Category) [5, definition 4.1]

A subset M of a metric space X is said to be:)

i. Rare (or nowhere dense) in X if its closure $\bar{M}$ has no interior points.
ii. Meager (or of the first category) in X if M is the union of countably many sets each of which is rare in X
iii. Nonmeager (or of second category) in X if M is not meager in X.

Definition 1.1 (Interior of a subset) *[6, definition 2.0]*

If A is a subset of a space X, the interior of A is the union of all open sets of X that are contained in A.

Definition 1.2 (Empty interior) *[7, definition 1.0]*

To say that A has empty interior is to say that A contains no open set of X other than the empty set.

Definition 1.3 (Baire space) *[8, definition 2.1]*

A space X is said to be Baire space if the following conditions hold: Given any countable collection $\{A_n\}$ of closed sets of X, each of which has empty interior in X, their union $\cup A_n$ also has empty interior in X.

Definition 1.4 (Metric space) *[9, definition 4.3]*

A metric space is a pair (X, d) where X is a set and d is a metric on X (or distance function on X), that is a function defined on XxX such that for every x, y, z, in X we have:

i. d is real-valued, finite and non-negative

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ii. d (x,y)=0 iff x=y
iii. d (x,y)=d (y, x) (symmetry)
iv. d (x,y) ≤d (x ,z)+d(z, y) (triangle inequality)

Definition1.5 (compactness) *[10, definition 1.0]*

A metric space X is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X, that is every sequence in M has a convergent subsequence whose limit is an element of M.

Definition1.6 (Local compactness) *[11, definition 3.2]*

A metric space X is said to be locally compact if every point of X has a compact neighborhood.

Definition 1.7 (Hausdorff space) *[12, definition 3]*

A topological space X is called a Hausdorff space if for each pair x_1, x_2 of disjoint points of X, there exists neighbourhoods $U, and\ U_2$ of $x_1\ and\ x_2$ respectively that are disjoint.

Theorem 1.8.0

If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.

Theorem 1.8.1 (Baire category theorem-Functional Analysis) *[13, theorem 1.1]*

If a metric space X≠∅ is complete, it is non-meager in itself. Hence if X≠∅ is complete and $X = \cup_{k=1}^{\infty} A_k, (A_k, closed)$then atleast one $A_k,$ contains a non-empty open subset.

Theorem1.8.2 (BCT1-General topology)

Every complete metric space is a Baire space. More generally, every topological space is homeomorphic to an open subset of a complete pseidometric space is a Baire space. Thus every completely metrixable topological space is a Baire space.

Theorem 1.8.3 (BCT2-General topology)

Every locally compact Hausdorff space is a Baire space

Theorem 1.8.4 (Baire category)

Suppose that X is a complete metric space and that $A_1, A_2, A_3,$ ....is a sequence of dense open sets. Then $\cap_{n=1}^{\infty} A_n,$is non-empty.

Theorem1.8.5 (Baire category)

Suppose that X is a complete metric space and that F_1, F_2, F_3 ... are closed sets, none containing any open Ball, $B_{\varepsilon,}(x)$.Then $\cup_{n=1}^{\infty} F_n,$ is the whole of X.

Theorem 1.8.6

There is a continuous function $f: [0,1] \to \mathbb{R}$ which is not differentiable at any point in (0,1).

Theorem 1.8.7 (Open mapping theorem, Bounded inverse theorem)

A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence if T is bijective, T^{-1} is continuous and thus bounded.

Theorem 1.8.8 (Closed graph theorem)

Let X and Y be Banach spaces and $T: D(T) \to Y$ a closed linear operator, where D(T)$\subset X$. Then if D(T) is closed in X, the operator T is bounded.

Theorem 1.8.9 (Uniform Bounded theorem)

Let (T_n) be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed linear space Y such that $(\| T_n x \|)$ is bounded for every x in X, say $\| T_n x \| \leq C_{x,}(n = 1,2,..)$ where $C_{x,}$ is a real number. Then the sequence of the norms $\| T_n \|$ is bounded, that is there is a C such that $\| T_n \| \leq C,$ (n=1,2,…)

Theorem 1.9.0 (theorem of continuous mapping)

A mapping T of a metric space X into a metric space Y is continuous iff the inverse image of any open subset of Y is an open subset of X because T is open.

Theorem 1.9.1 (Theorem of inverse operator)

Let X and Y be vector spaces, both real or both convex. Let $T: D(T) \to Y$ be a linear operator with domain D(T)$\subset X$ and range R(T)$\subset Y$. Then

a. the inverse $T^{-1}: R(T) \to D(T)$ exists if Tx=0 implying x=0
b. If T^{-1}exists,it is a linear operator
c. If dimD(T)=n<∞ and T^{-1} exists, then dim R(T)=dim D(T)

Theorem 1.9.2 (Theorem of continuity and boundedness)

Let $T: D(T) \to Y$ be a linear operator, where D(T)$\subset X$and X,Y are normed linear spaces. Then:

a. T is continuous iff T is bounded
b. If T is continuous at a single point, it is continuous.

Results and discussions

Baire category theorem, theorem 1.8.2 is used in functional analysis to prove the following theorems: open mapping theorem 1.8.7, the closed graph theorem1.8.8 and the uniform boundedness theorem 1.8.9.

Proof of theorem 1.8.9 (Uniform boundedness theorem)

For every $k \in \mathbb{N}$, let $A_k \subset X$ be the set of all $x: \| T_n x \| \leq k \; \forall \, n$. A_k is closed. In deed for every x in the closure of A_k, there exists a sequence (x_j) in A_k covering x. This means that for every fixed n we have $\| T_n \, x_j \| \leq k$ and obtains $\| T_n x \| \leq k$ because T_n is continuous and so is the norm.
Since X is complete, Baire's theorem implies that some $A_{k,}$ contains an open ball, say $B_0 = B(x_0; r) \subset A_{k0}$. Let x$\subset X$ be arbitrary, not zero. We set $z = x_0 + \gamma x, \gamma = \frac{r}{2\|x\|}$.
Then $\| z - x_0 \| < r$, so that $z \in B_0$.This implies that we must have $\| T_n z \| \leq k_0 \, \forall n$. Also $\| T_n x_0 \| \leq k_0$ since $x_0 \in B_0$.This implies that $x = \frac{1}{\gamma}(z - r_0)$.This yields for all n.

$$\| T_n x \| = \frac{1}{\gamma} \| T_n (z - x_0) \| \leq \frac{1}{\gamma} (\| T_n z \| + T_n x_0) \leq \frac{4}{r} \| x \| k_0 .$$

Hence ∀ n , $\| T_n \| = \underbrace{Sup}_{\|x\|=1} \| T_n x \| \leq \frac{4}{r} k_0$ where $C = \frac{4}{r} k_0$.

Proof of open mapping theorem and Bounded inverse theorem

We prove that for every open set A$\subset X$ the image T(A) is open in Y. This we do by showing that ∀ y$\subset T_x \in T(A)$ the set T(A) contains an open ball about y $= T_x$. Let $y = T_x \in T(A)$ .Since A is open, it contains an open ball with centre x. Hence A-x contains an open ball with centre 0; let the radius of the ball be r and set $\frac{1}{r} = k$, so that $\frac{1}{k} = r$. Then k(A-x) contains the open unit ball B(0,1).Baire's category theorem now implies that T(k(A-x)=k[T(A)-Tx] contains an open ball about $T_x = y$. Since y =T(A) was arbitrary, T(A) is open. Finally if $T^{-1}: Y \rightarrow$X exists, it is continuous by the theorem of continuous mapping, theorem 1.9.0.By the theorem of inverse operator, theorem 1.9.1, T^{-1} is linear and it is bounded by the theorem of continuity and boundedness, theorem 1.9.2

Baire category theorem also shows that every complete metric space with no isolated points is uncountable. (If X is a countable complete metric space with no isolated points, then each singleton {x} in X is nowhere dense and so X is of first category in itself) in particular, this proves that the set of all real numbers is uncountable.

Baire's category theorem shows that each of the following is a Baire space:

i. The space ℝ of real numbers
ii. The irrational numbers, with the metric defined by $d(x,y) = \frac{1}{n+1}$,where n is the first index for which the continued fraction expansions of x and y differ(T2 is a complete metric space)
iii. The cantor set is a Baire space.

In Topology, by Baire's category theorem, theorem 1.8.3, every finite-dimensional Hausdorff manifold is a Baire space, since it is locally compact and Hausdorff. This is so even for a non- paracompact (hence non metrizable) manifolds such as the long line

Conclusions

The proof of Baire-category theorem made it possible to derive from it the uniform boundedness theorem as well as the open mapping theorem. Infact throughout analysis there are many instances of results related to this theorem. In this paper, we can see how the Baire's category theorem resonates through the general topology and functional analysis, making it an important tool in various proofs of theorems.

Conflicts of interest

Authors declare no conflict of interest.

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