Math 3331 – ODE's

First Order ODE's

$$\frac{dy}{dx} = F(x,y)$$

So far

(1) Separable $\quad \frac{dy}{dx} = f(x)g(y)$

so $\quad \frac{dy}{g(y)} = f(x)dx$

and integrate $\quad \int \frac{dy}{g(y)} = \int f(x)dx + C$

(2) Linear

$$\frac{dy}{dx} + p(x)y = q(x)$$

integrating factor $\quad \mu = e^{\int p(x)dx}$

so $\quad \mu\left(\frac{dy}{dx} + p(x)y\right) = \mu q$

$\Rightarrow \quad \frac{d}{dx}(\mu y) = \mu q \quad \leftarrow$ sep & integrate

(3) Bernoulli

$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad n \neq 0, 1$$

Divide by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + p(x)\frac{y}{y^n} = q(x)$$

let $u = \frac{y}{y^n}$

turn ODE into one that is linear

Ex $\quad \frac{dy}{dx} + y = xy^3$

divide by y^3 so $\quad \frac{1}{y^3}\frac{dy}{dx} + \frac{y}{y^3} = x$

let $u = \frac{1}{y^2} \quad \frac{du}{dx} = -2y^{-3}\frac{dy}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$

so $\quad -\frac{1}{2}\frac{du}{dx} + u = x \Rightarrow \frac{du}{dx} - 2u = -2x$ linear ST

$\mu = e^{-2x} \quad \frac{d}{dx}(e^{-2x}u) = -2xe^{-2x}$

$e^{-2x}u = \frac{(2x+1)}{2}e^{-2x} + c \Rightarrow \boxed{\frac{1}{y^2} = \frac{2x+1}{2} + ce^{2x}}$ Sol^n

(4) <u>Homogeneous</u>

ODEs of the form

$$\frac{dy}{dx} = f(y/x)$$

<u>Ex</u> $\frac{dy}{dx} = \frac{y}{x}$ Sep

$\frac{dy}{dx} = \frac{y}{x} + 1$ linear

$\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$ Bernoulli

$\frac{dy}{dx} = \frac{x^2+y^2}{x^2} = 1 + (y/x)^2$ none of the above.

<u>Idea.</u> let $u = y/x$ so $f(y/x) = f(u)$

so $y = xu$

$\frac{dy}{dx} = \frac{d}{dx}(xu) = x\frac{du}{dx} + u$

ODE becomes $x\frac{du}{dx} + u = f(u)$ ← now sep

Ex1 $\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1$

$y = xu \qquad \frac{dy}{dx} = x\frac{du}{dx} + u$

so $\qquad x\frac{du}{dx} + \cancel{u} = 1 + \cancel{u} + u^2$

$\int \frac{du}{1+u^2} = \int \frac{dx}{x} \qquad \tan^{-1} u = \ln|x| + c$

$\tan^{-1}(y/x) = \ln|x| + c$

so $\qquad y/x = \tan(\ln|x| + c)$

Ex2 $\qquad \frac{dy}{dx} = \frac{x-y}{x+y}$

Is this homogeneous? is

$f(x,y) = \frac{x-y}{x+y} = f(y/x)$? $\qquad$ Hard Questn if we replace $x \to \lambda x$, $y \to \lambda y$

in $F(x,y)$ and we get

$F(\lambda x, \lambda y) = F(x,y)$ then yes

Ex $F(x,y) = \dfrac{x-y}{x+y}$

$$F(\lambda x, \lambda y) = \frac{\lambda x - \lambda y}{\lambda x + \lambda y} = \frac{\cancel{\lambda}(x-y)}{\cancel{\lambda}(x+y)} \checkmark$$

so it is homogeneous

$y = xu \quad \dfrac{dy}{dx} = x\dfrac{du}{dx} + u$

$$x\frac{du}{dx} + u = \frac{x - xu}{x + xu} = \frac{\cancel{x}(1-u)}{\cancel{x}(1+u)}$$

$$x\frac{du}{dx} = \frac{1-u}{1+u} - u$$

$$= \frac{1-u-u(1+u)}{1+u} = \frac{1-2u-u^2}{1+u}$$ separable

so $\dfrac{u+1}{u^2+2u-1}\,du = -\dfrac{dx}{x}$

$v = u^2+2u-1$

$dv = (2u+2)du \Rightarrow \dfrac{dv}{2v} = -\dfrac{dx}{x}$

so $\dfrac{1}{2}\ln|v| = -\ln|x| + \frac{1}{2}\ln c$

$v = \dfrac{c}{x^2}$

so $u^2+2u-1 = \dfrac{c}{x^2}$

so $\left(\dfrac{y}{x}\right)^2 + \dfrac{2y}{x} - 1 = \dfrac{c}{x^2}$

or mult by x^2

$y^2 + 2xy - x^2 = c$

I would stop here but we could go farther

$(y+x)^2 = c+2x^2 \qquad y = -x \pm\sqrt{c-2x^2}$