Simulating the Feasible Set and Efficient Frontier

Charles J. Higgins

Abstract

While the theoretical graphical presentation for development of the Capital Asset Pricing Model is replete throughout financial literature and includes the feasible set and efficient frontier, a simulation is rarer and serves to demonstrate that it is not only efficacious but also serves to present the most counter intuitive and profound aspects found therein.

Keywords: CAPM, Efficient Frontier, security analysis, Markowitz theory, EMH

Introduction

One of basic tenets developed in finance is the Capital Asset Pricing Model (CAPM) and is replete throughout the literature; see the best selling college/university investments text Bodie, Kane, & Marcus (2001, 2010, & 2017) and advanced text Elton & Gruber (1995 and later). In particular, it converted the analysis of risk from anecdotal to numeric. Likewise, it commences with the number of securities *n*, which combine into portfolios with a return of:

$$R_p = \sum_{i=1}^{n} w_i R_i$$

where R_p is the return to the portfolio, R_i is the return of security *i*, and the weight w_i represents the proportion to the whole portfolio, given that:

$$\sum_{i=1}^{n} w_i = 1.$$

Note that some weights may be negative reflecting a borrowed short position. The risk of a portfolio is measured by its standard deviation, the square root of the portfolio's variance or:

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

where σ_p^2 is the variance of the portfolio and σ_{ij} is the covariance of the security *i* by *j*. Markowitz [1959] examined a two-space of vertical returns and horizontal risk measured by the standard deviation of the returns with the preference toward higher/ upward returns and toward lesser/leftward risk. In examining the risk of a portfolio comprised of two securities, the previous equation measuring portfolio risk becomes:

$$\sigma_p^2 = w_a^2\sigma_a^2 + w_b^2\sigma_b^2 + 2\, w_a w_b \sigma_{ab}$$

where σ_{ab} also equals $\sigma_a\sigma_b\rho_{ab}$ where ρ_{ab} is the correlation coefficient. Markowitz noted that in the risk-return space that all theoretically feasible portfolios create either a straight line or a curved line to left, or:

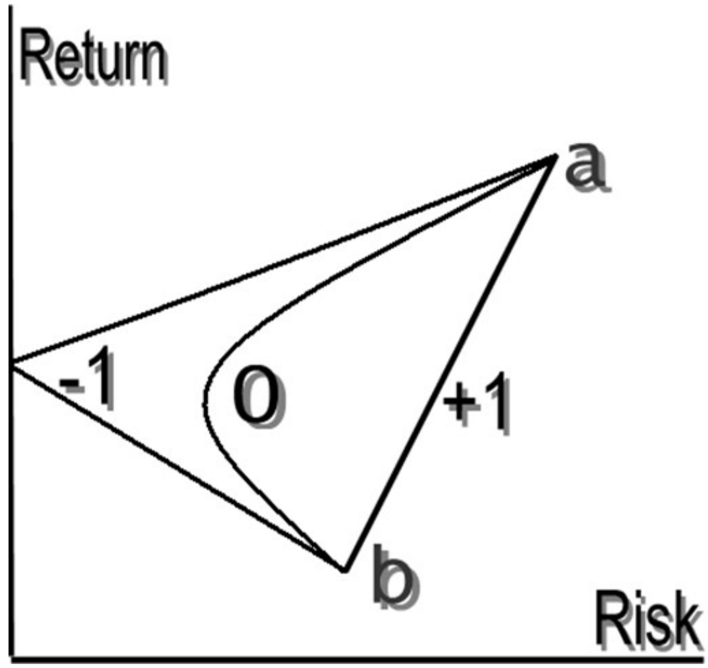

Figure 1: Feasible Portfolios with Correlations

A third security can be added to a given previously weighted portfolio, and so on, until a Feasible Set of portfolio choices exists, and Markowitz showed that the shape or envelope of the Feasible Set would be continuously smooth on its left side, or:

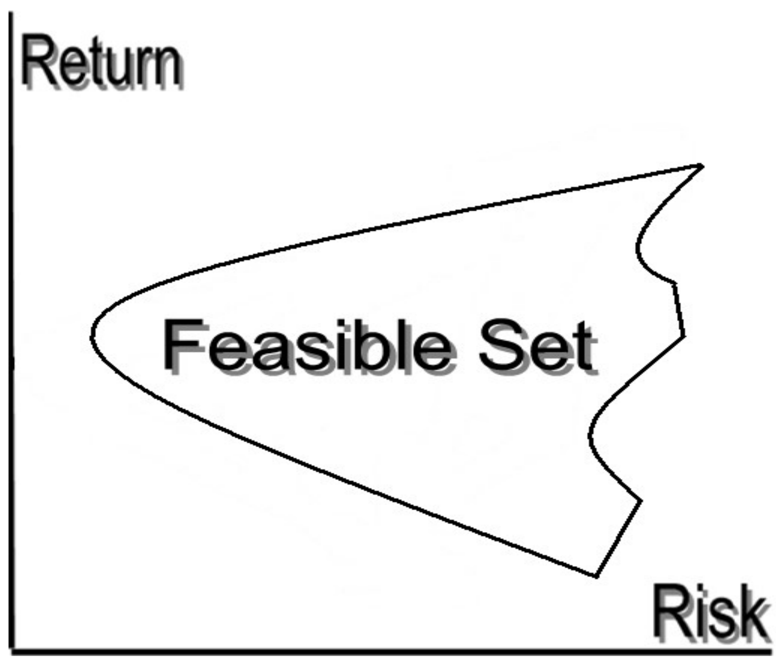

Figure 2: The Feasible set of Risky Securities

If a risk-free security is also introduced, the portfolio variance equation simplifies and becomes both linear and directly proportional, or $\sigma_p = w_a\sigma_a$ with a as the risky security. The addition of the risk-free security adds the possibility of a straight Capital Market Line (and hence the Capital Asset Pricing Model) from the risk-free security to the optimal tangency on the Feasible Set given a preference for a higher return for any given amount of risk, or:

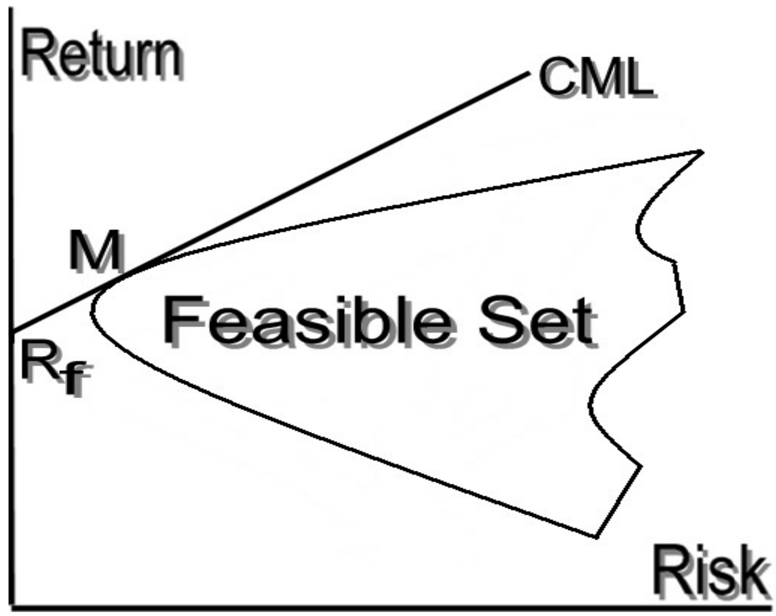

Figure 3: Risk-free Security Portfolio Choices on the CML

It is argued that the tangency should be that portfolio which reflects the risky Market as a whole. However, Roll [1977] has demonstrated that one cannot prove nor disprove that the tangency is indeed the Market. Note also that the Capital Market Line to the right of the Market tangency is where the weight of the risk-free security is negative and thus is often described as the borrowing region of the Capital Market Line. It now follows that any portfolio choice optimizing return and minimizing risk lies on the Capital Market Line and is composed uniquely of only two choices for *all* investors—the risk-free security and the Market. Any rational risk adverse investor must choose only among the risk-free and market index at the tangency point. Of course, each investor chooses his/her appropriate mix of these two, and that any other set of choices creates an inferior portfolio in terms of return and risk.

A more effective argument can be found in simulating the feasible set and selecting those weights which produce portfolios on the efficient frontier near a supposed Capital Market Line. To do so, I wrote a Basic program producing this effective graphic among many others:

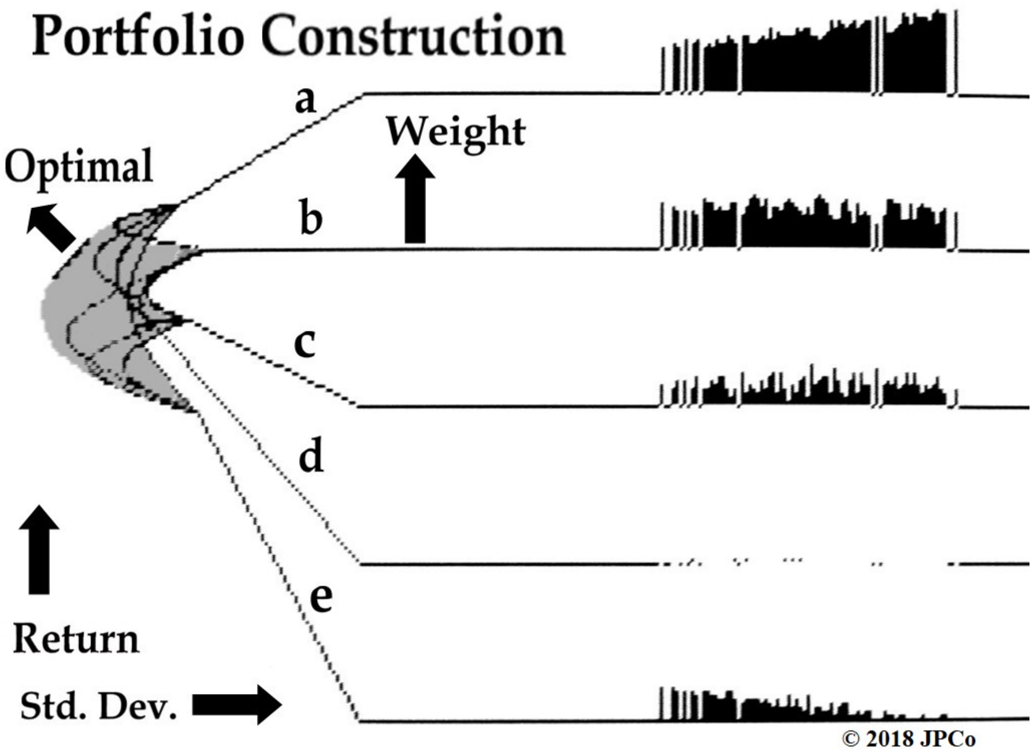

Figure 4: Simulated feasible set weights for 5 securities

This graphic particularly demonstrates that not only is the feasible set smooth and continuous bending to the left, but that the weightings that produce portfolios along the efficient frontier that are upward and to the left include a security that has the worst return and risk (here e)!
To produce this graphic, the following Basic program was created:

```
10  RANDOMIZE
20 SCREEN 1,0:COLOR 0,1
30 LINE (0,0)-(400,200),3,BF
40 ZZ=0
50 DY=.8
60 XL=320
70 YL=190
80 G=999
90 N=4
100 DD=YL/(N+1)
110 M=6
120 DIM R(N,M),A(N),S(N),P(M),W(N)
130 FOR I=0 TO N
140 A(I)=0
150 S(I)=0
160 FOR J=1 TO M
170 R(I,J)=RND(1)*YL
180 A(I)=A(I)+R(I,J)
190 NEXT J
200 A(I)=A(I)/M
210 NEXT I
220 K=0
230 FOR I=0 TO N-1
240 IF A(I)>A(I+1) THEN 290
250 FOR J=1 TO M
260 X=R(I,J):R(I,J)=R(I+1,J):R(I+1,J)=X
270 NEXT J:K=1
280 X=A(I):A(I)=A(I+1):A(I+1)=X
290 NEXT I
292 IF K=1 THEN 220
300 FOR I=0 TO N
310 FOR J=1 TO M
320 D=ABS(A(I)-R(I,J))
330 S(I)=S(I)+D*D
340 NEXT J
350 S(I)=SQR(S(I)/M)
360 LINE (S(I),YL-A(I))-(100,I*DD+DD),0
370 LINE (XL,I*DD+DD)-(100,I*DD+DD),0
380 NEXT I
390 T=0
400 CY=1
410 FOR I=0 TO N-1
420 W(I)=RND(1):W(I)=W(I)*W(I)
430 IF W(I)>DY THEN CY=2
440 T=T+W(I) :IF T>1 THEN 390
450 NEXT I
460 W(N)=1-T :IF W(N)<0 OR W(N)>1 THEN 390
470 FOR J=1 TO M
480 IF W(N)>DY THEN CY=2
490 V=0:P(J)=0
500 FOR I=0 TO N
510 V=V+R(I,J)*W(I)
520 NEXT I
530 P(J)=V
540 NEXT J
```

```
550 A=0
560 S=0
570 FOR J=1 TO M
580 A=A+P(J)
590 NEXT J
600 A=A/M
610 FOR J=1 TO M
620 D=ABS(A-P(J))
630 S=S+D*D
640 NEXT J
650 S=SQR(S/M)
660 SS=7*S+60
670 IF S+(YL-A)-Z<=G THEN 700
680 PSET (S,YL-A),CY
690 GOTO 390
700 IF S+(YL-A)<G THEN G=S+(YL-A)
710 ZZ=ZZ+1:IF ZZ<20 THEN 390
720 FOR I=0 TO N:J=I*DD+DD
730 C=W(I)*N*DD/4
740 LINE (SS,J)-(SS,J-C),0
750 LINE (SS,J-C)-(SS,J-DD),3
760 LINE (S(I),YL-A(I))-(100,I*DD+DD),0
770 NEXT I
780 PSET (S,YL-A),0
790 GOTO 390
Ok

1LIST  2RUN←  3LOAD"  4SAVE"  5CONT←  6,"LPT1  7TRON←  8TROFF←  9KEY  0SCREEN
```

Figure 5: Program simulating the feasible set & efficient weights

In this program written in GWBasic required the download of DOSBox for computers having 32-bit or 64-bit programs. Lines numbered 10 through 30 initialize the program to a higher graphic resolution and "blacks out" the screen. Thereafter lines through 110 the size of the screen plus 5 securities with 6 random returns to be determined are specified. Through line 210 the random returns are created and their respective means are calculated. Through 292 the data are sorted for better graphic presentation. Through 350 the standard deviation of each security's return is calculated. Lines 360 and 370 plot security's coordinates and draws a connecting line to where the weights will be drawn. Lines 390 through 460 create random weights assuring that the total weights add to 1. Lines through 540 compute the portfolio return and then through 650 compute the portfolio standard deviation. Lines 670 and 680 plot the portfolio if it is interior to the feasible set—but if otherwise branches to line 700. Starting at 700 the portfolio has been judged to be very close to the upward and leftward limit (highest return and least standard deviation) along the efficient frontier and the weights are also plotted between each security's weighting range between 0 and 1. Likewise, each security pair inside the feasible set is also plotted.

Conclusion

I have found this graphic program (along with the extreme counter intuitive prior exemplar) to be particularly helpful in presenting the derivation of the Capital Asset Pricing Model. I like to note that the addition of the worst return and worst risk security in the prior exemplar with a relatively low weighting can be viewed as "adding pepper to soup—not too much, but very beneficial." Of course other simulation runs can be less profound. Some do not differentiate the returns very well, others do not produce a realistic Efficient Frontier when either the correlations are high or when the return and risk are highly dominated (little conflict in hierarchical ordering).

References

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Author

Charles J. Higgins
Professor of Finance, Department of Finance/CIS, Loyola Marymount University, USA, chiggins@lmu.edu