Math 1496 - Sample Test 2

1. If $f(x) = x^3 + x + 1$ find $(f^{-1})'(3)$.

Soln. The inverse is given by $x = y^3 + y + 1$. We wish to find the derivative at $x = 3$ so we need the y value. Therefore, $3 = y^3 + y + 1$ or $y^3 + y - 2 = 0$ or $(y-1)(y^2 + y + 2) = 0$ giving that $y = 1$. Since $x = y^3 + y + 1$ then $1 = (3y^2 + 1)y'$ or $y' = \dfrac{1}{3y^2 + 1}$ and at $y = 1$ $y' = 1/4$.

2. Find the absolute minimium and maximum of the following on the given interval

$$\begin{array}{cc} (i) & f(x) = 1 - x^2 \quad \text{on} \ \ [-2,3] \\ (ii) & f(x) = 2x^3 - 15x^2 + 24x \quad \text{on} \ \ [0,3] \end{array}$$

Soln (i). Since f is continuous on $[-2,3]$ and differentiable on $(-2,3)$ it will have a minimum and maximum. These will be located at the endpoints or inside the interval at the critical points. So here $f' = -2x$ and $f' = 0$ when $x = 0$.

$$f(-2) = -3, \quad f(0) = 1(max), \quad f(3) = -8(min). \tag{1}$$

Soln (ii). Since f is continuous on $[0,3]$ and differentiable on $(0,3)$ it will have a minimum and maximum. These will be located at the endpoints or inside the interval at the critical points. So here $f' = 6x^2 - 30x + 24 = 6(x-1)(x-4)$ and $f' = 0$ when $x = 1, 4$ but $x = 4$ is outside the interval

$$f(0) = 0, \quad f(1) = 11(max), \quad f(3) = -9(min). \tag{2}$$

3. State the Mean Value Theorem. Verify the Mean Value Theorem for the following

$$\begin{array}{cc} (i) & f(x) = x^3 - x \quad \text{on} \ \ [0,2] \\ (ii) & f(x) = \dfrac{x}{x+2} \quad \text{on} \ \ [1,10] \end{array}$$

Soln. The MVT states that if $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then there exists a c (at least one) in (a,b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{3}$$

Soln (i) We have $\dfrac{f(2) - f(0)}{2 - 0} = \dfrac{6}{2} = 3$. Also $f' = 3x^2 - 1$ and $f'(c) = 3c^2 - 1 = 3$ gives $c = 2/\sqrt{3}$

Soln (ii) We have $\dfrac{f(10) - f(1)}{10 - 1} = \dfrac{1/2}{9} = 1/18$. Also $f'(x) = \dfrac{2}{(x+2)^2}$ and

$$f'(c) = \frac{2}{(c+2)^2} = \frac{1}{18} \quad \Rightarrow \quad c = -8, 4$$

from which we choose $c = 4$.

4. If $y = x(x-4)^3$ calculate the following

(i) The critical numbers

(ii) When y is increasing and decreasing.

(iii) Determine whether any of the critical numbers are minimum or maximum.

(iv) When y is concave up and down and determine the points of inflection.

(v) Then sketch the curve.

Soln

$$y' = (x-4)^3 + 3x(x-4)^2 = 4(x-1)(x-4)^2$$

and $y' = 0$ when $x = 1, 4$ (Critical numbers)

$$y'' = 4(x-2)^2 + 8(x-1)(x-4) = 12(x-2)(x-4)$$

and $y'' = 0$ when $x = 2, 4$ (possible PI's.)

x		1		2		4	
$x-1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-4)^2$	$+$	$+$	$+$	$+$	$+$	0	$+$
$(x-1)(x-4)^2$	$-$	0	$+$	$+$	$+$	0	$+$
slope	$\backslash$	$\_\_$	$/$	$/$	$/$	$-$	$/$
$x-2$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x-2)(x-4)$	$+$	$+$	$+$	0	$-$	0	$+$
h/v	$\smile$	$\smile$	$\smile$	PI	$\frown$	PI	$\smile$

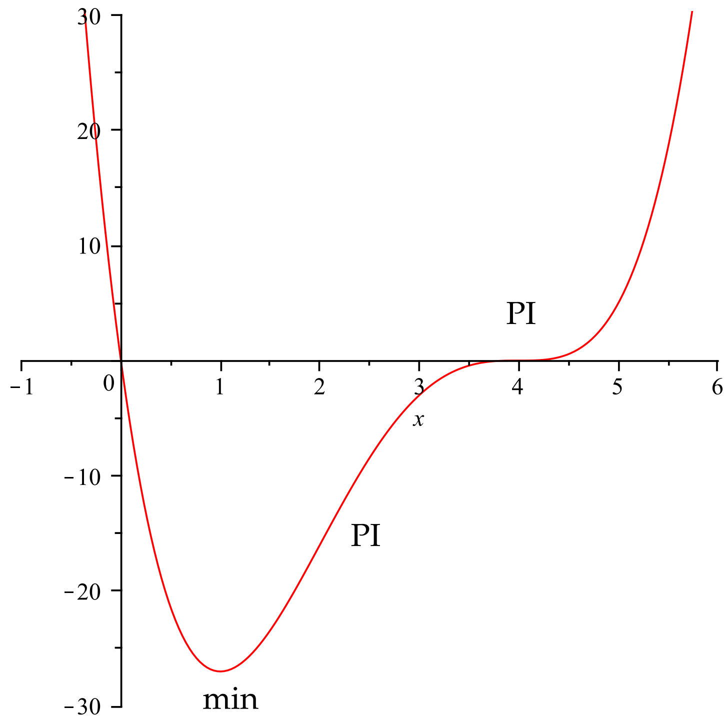
critical numbers $x = 1, 4$

increasing $(1,4),\ (4,\infty)$ decreasing $(-\infty, 1)$

$\min(1, -27)$ max - none

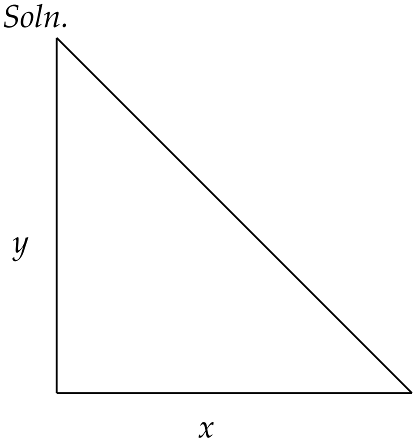
concave up $(\infty, 2),\ (4, \infty)$ concave down $(2, 4)$

$\mathrm{PI}(2, -16), (4, 0)$



5. A ladder 13 feet long is resting against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 ft/sec. At rate is the tip of the ladder moving down the wall when the base of the ladder is 5 ft away from the wall?

Soln.



What we know: $\dfrac{dx}{dt} = 2$

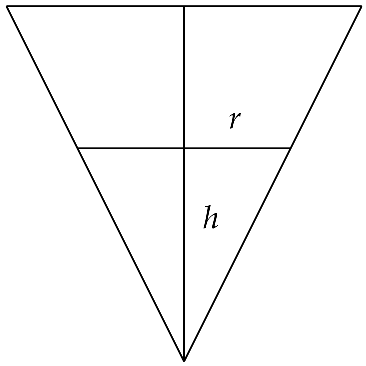
What we want: $\dfrac{dy}{dt}$ when $x = 5$

Relate variables: $x^2 + y^2 = 13^2$

Relate rates: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$ so $\dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt}$

When $x = 4, y = 12$ so $\dfrac{dy}{dt} = -\dfrac{5}{12} \cdot 2 = -\dfrac{5}{6}$ ft/sec.

6. A paper cup in the shape of an inverted cone with height 10 cm and a base of radius 3 cm, is being filled at a rate of 2 cm^3/min. Find the rate of change in the height of the water when the height of the water is 5 cm.



What we know: $\dfrac{dV}{dt} = 2$

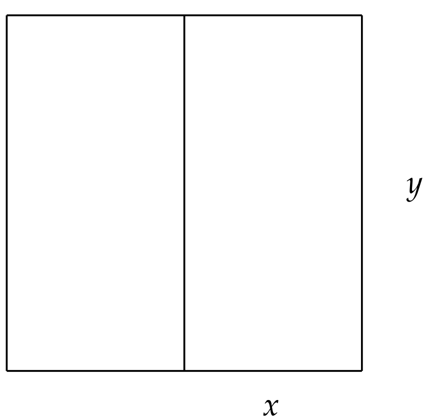
What we want: $\dfrac{dh}{dt}$ when $h = 5$

Relate variables: $V = \dfrac{1}{3}\pi r^2 h$. We also have similar triangles so $\dfrac{h}{10} = \dfrac{r}{3}$ so $V = \dfrac{3\pi h^3}{100}$

Relate rates: $\dfrac{dV}{dt} = \dfrac{9\pi h^2}{100}\dfrac{dh}{dt} = 0$ so $\dfrac{dh}{dt} = \dfrac{100}{9\pi h^2}\dfrac{dV}{dt}$

When $h = 5$ $\dfrac{dh}{dt} = \dfrac{100}{9\pi 5^2} \cdot 2 = \dfrac{8}{9\pi}$ cm/min.

7. A rancher has 200 feet of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?



Soln. Once we draw and label we want to maximize $A = 2xy$ subject to $4x + 3y = 200$ Isolating y gives

$$y = \frac{200 - 4x}{3}$$

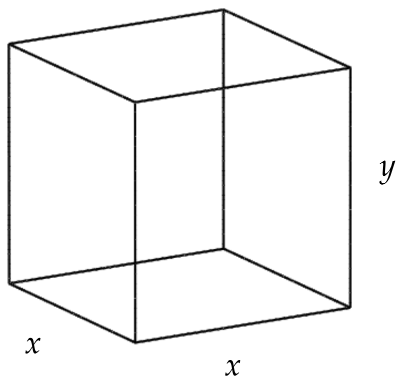
so

$$A = \frac{2x(200 - 4x)}{3} = \frac{400x - 8x^2}{3}$$

$$A' = \frac{400 - 16x}{3}$$

$A' = 0$ when $x = 25$ and $A'' = -16/3 < 0$ so a max. Each pen is $25' \times 100/3'$.

8. An box with a square bottom is to be built that holds 24 cubic feet. Find the dimensions of the box that will minimize the surface area of the box.



We label the box with base x and x (it has a square base) and height y. The volume is therefore

$$V = x^2 y.$$

The surface area A is

$$A = 2x^2 + 4xy = 24.$$

We solve for y giving

$$y = \frac{24 - 2x^2}{4x} = \frac{12 - x^2}{2x}$$

and so V is

$$V = x^2 \frac{12 - x^2}{2x} = \frac{12x - x^3}{2}$$

$$V' = \frac{12 - 3x^2}{2}$$

Solving $V' = 0$ gives $x = \pm 2$ from which we choose $x = 2$. Now $V'' = -3x < 0$ for $x = 2$ giving a maximum. With $x = 2$ we solve for y giving $y = 2$ and the dimensions of the box is $2' \times 2' \times 2'$.