Resilient Vector Consensus using Centerpoint

M. Shabbir[1], J. Li[2], W. Abbas[2], and X. Koutsoukos[2]

[1]*Information Technology University, Lahore, Pakistan*

[2]*Vanderbilt University, Nashville, TN*

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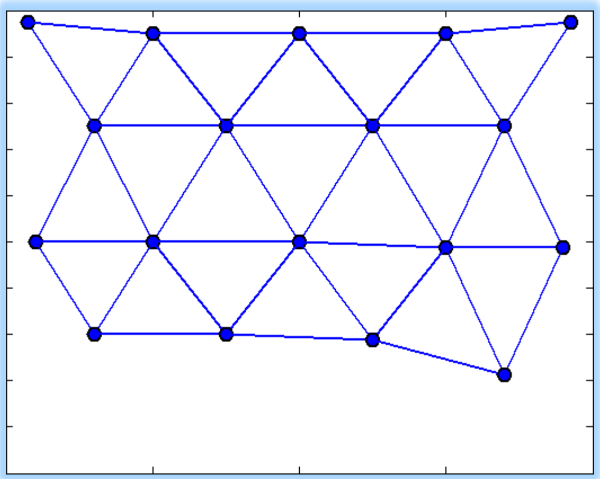
Vector Consensus

A network of agents modeled by an undirected graph $G = (V,E)$.

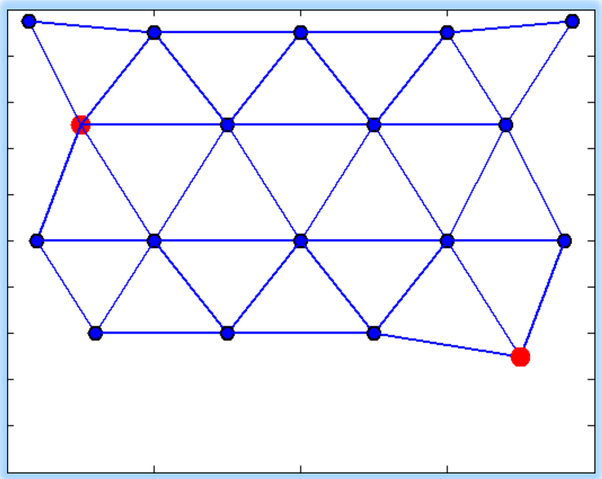
State of agent *i* at time *t* is $x_i(t) \in \mathbb{R}^d$, where $d \geq 2$.

Applications

- Control of moving **vehicles (UAVs)**
- Information processing in **sensor networks**
- Design of **distributed optimization** algorithms
- Parameter **estimation** etc.



Consensus



No Consensus due to **adversaries**

How can we design a ***resilient vector consensus*** algorithm?

Resilient Vector Consensus

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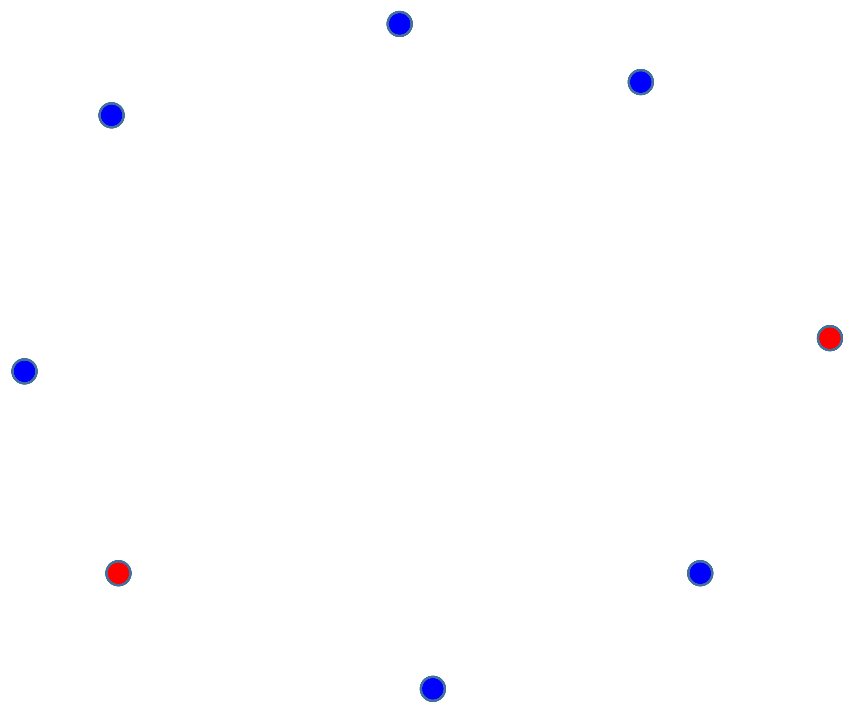

Agents' initial positions

Blue → normal

Red → adversary

Resilient Vector Consensus

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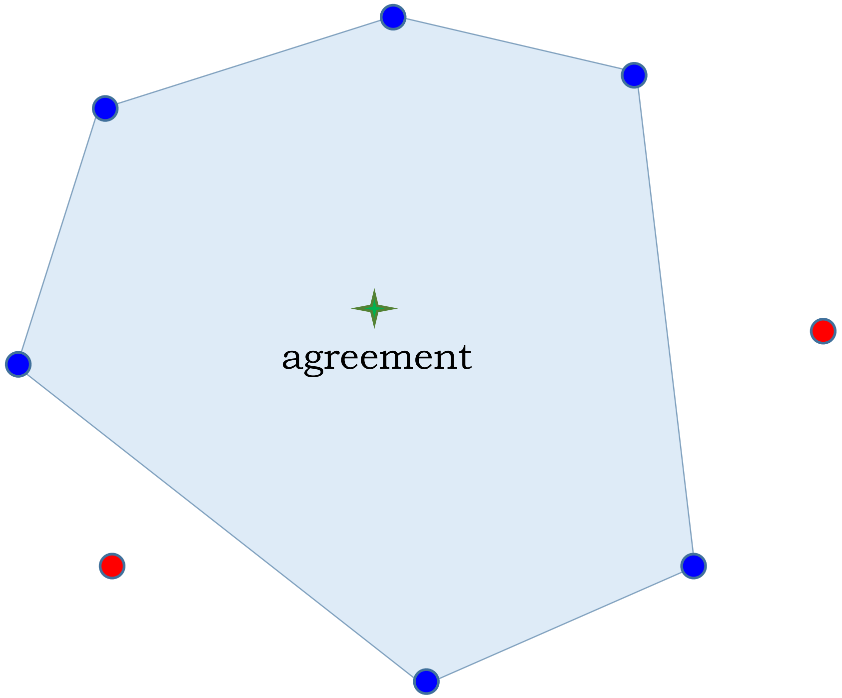


Convex hull of normal agents' initial positions

Safety:

At all times, every normal agent should *remain inside the convex hull* of all normal agents' initial positions.

Agreement:

All normal agents should eventually *converge* at a common point.

Blue → normal

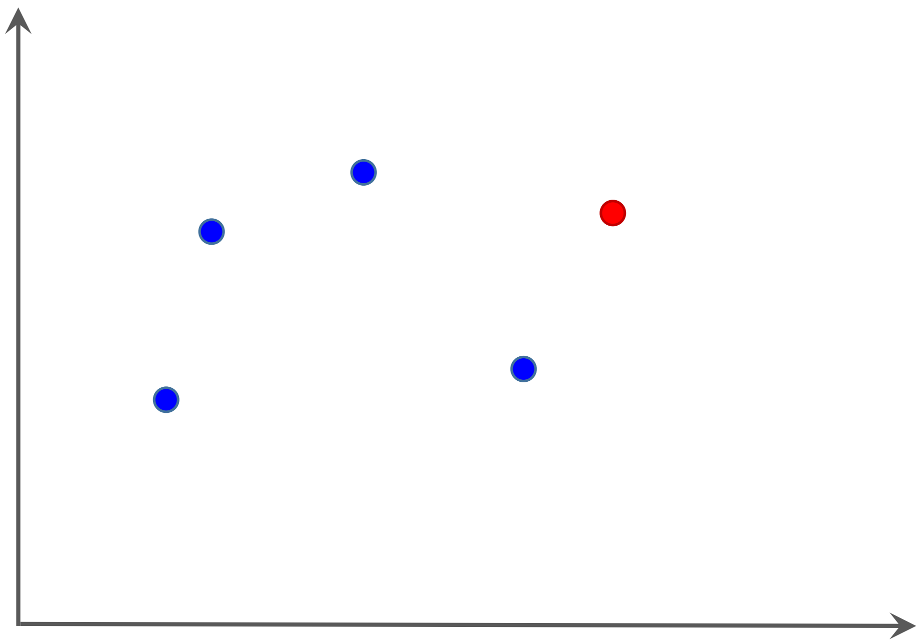
Red → adversary

Can We Use Resilient Scalar Consensus?

There are well studied resilient scalar consensus ($x_i(t) \in \mathbb{R}$) algorithms.[1]

First Approach:

Implement scalar resilient consensus algorithm in each dimension separately.



[1] H. LeBlanc, H. Zhang, X. Koutsoukos, and S. Sundaram, "Resilient asymptotic consensus in robust networks," *IEEE J Sel. Areas Comm., 2013.*

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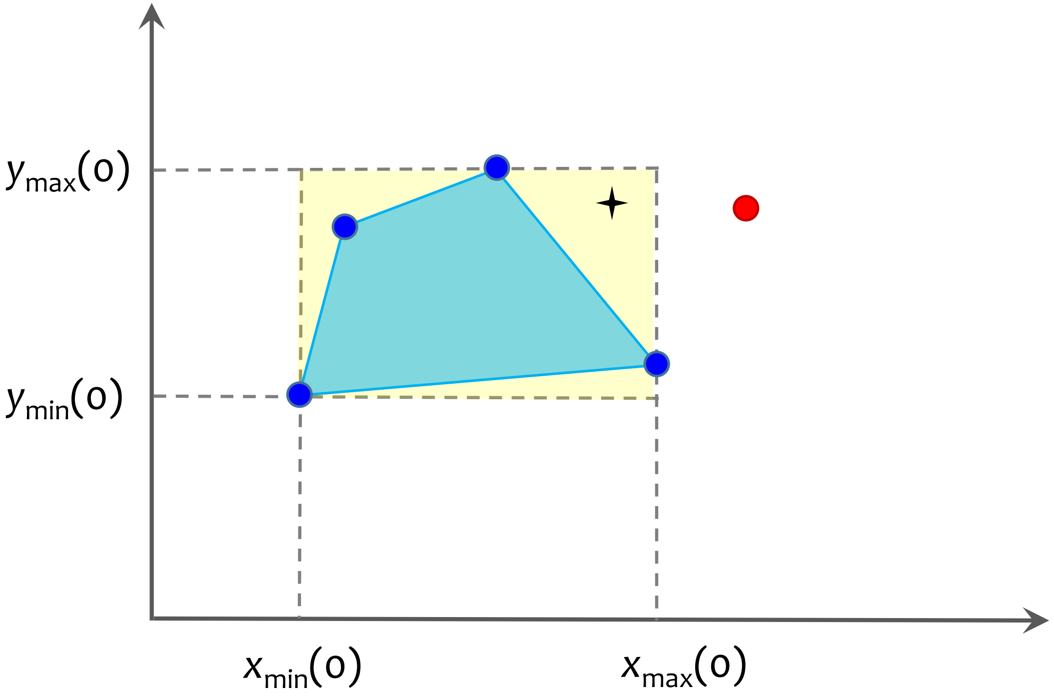


Normal agents can end up converging outside of the convex hull of their initial positions.

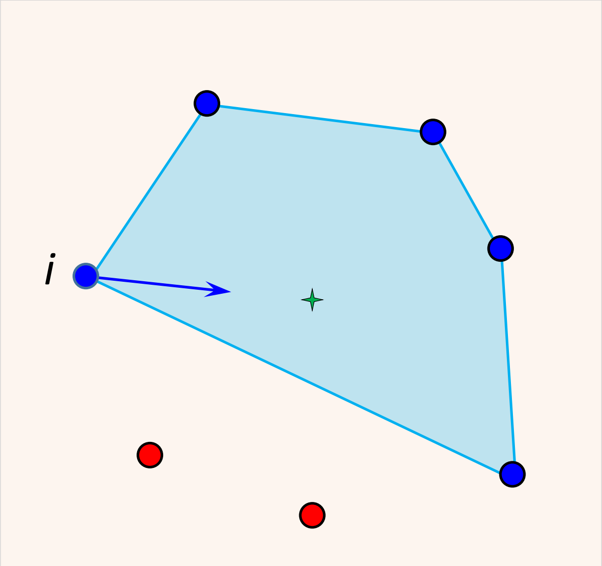
Implementing multiple instances of scalar resilient consensus does not work.

[1] H. LeBlanc, H. Zhang, X. Koutsoukos, and S. Sundaram, "Resilient asymptotic consensus in robust networks," *IEEE J Sel. Areas Comm.*, 2013.

Approximate Distributed Robust Convergence (ADRC)

ADRC is resilient vector consensus algorithm proposed by Park and Hutchinson.[2]

1. In each iteration *t*, a normal agent *i* finds a point $s_i(t)$ that lies in the convex hull of its normal neighbors' states.
2. Agent *i* updates its state by moving towards $s_i(t)$.



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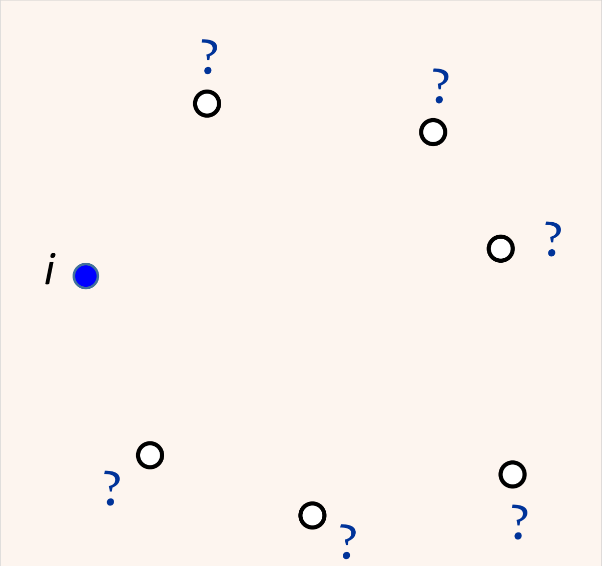


Challenge:

A normal agent doesn't know who is normal/adversary in its neighborhood.

Safe point

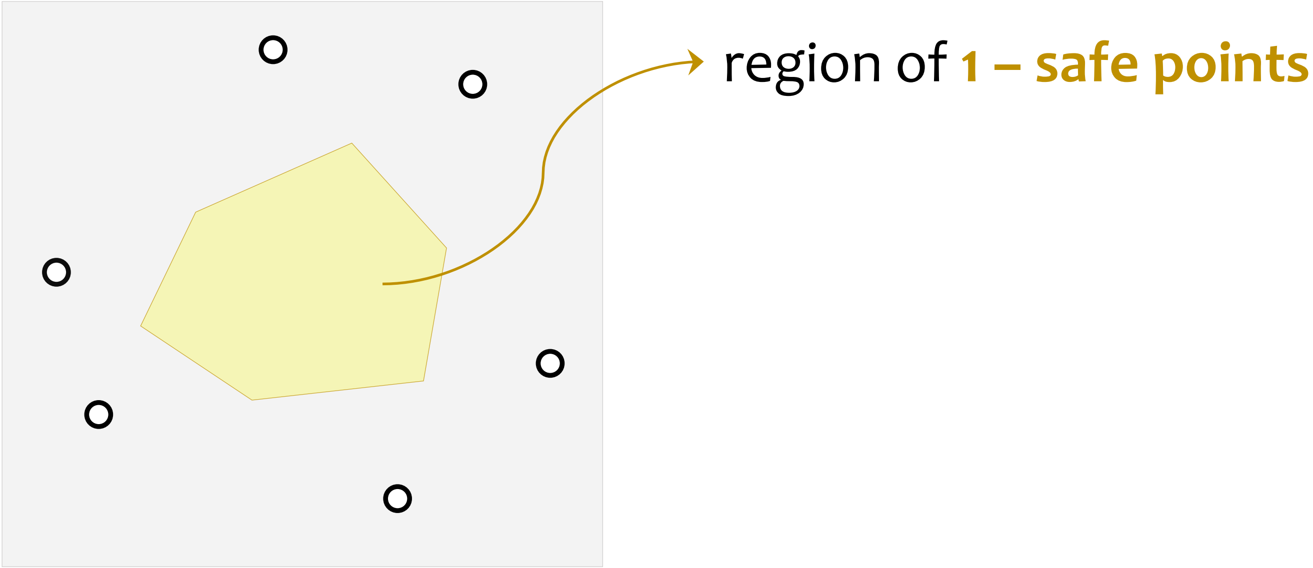
[2]H. Park and S. Hutchinson, "Fault-tolerant rendezvous of multirobot systems," *IEEE Trans. Robotics, 2017.*

F – Safe Points

F – Safe Point:

Given a set of *N* points in R^d, of which *any* of the *F* points can be adversarial (corresponding to adversarial agents).

Then, a point that is guaranteed to lie in the convex hull of (*N* – *F*) normal points is an *F* – Safe point.



N = 6, d = 2, F = 1

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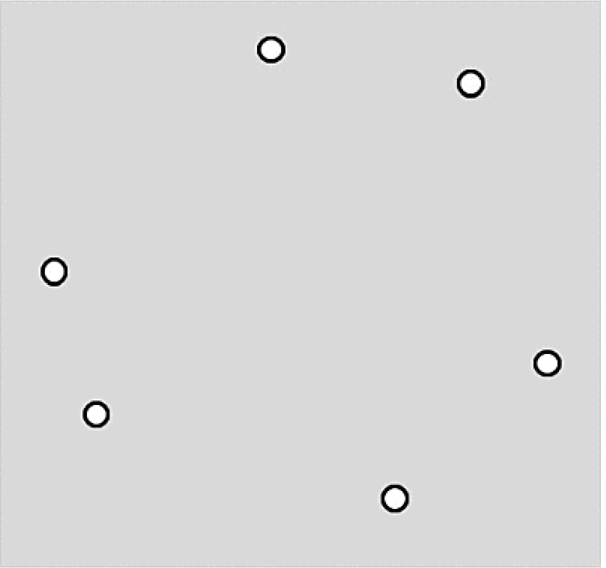

N = 6, d = 2, F = 1

- normal
- adversary

1 – safe region (yellow) always lies in the **convex hull (blue)** of normal nodes, regardless of the selection of the adversary node.

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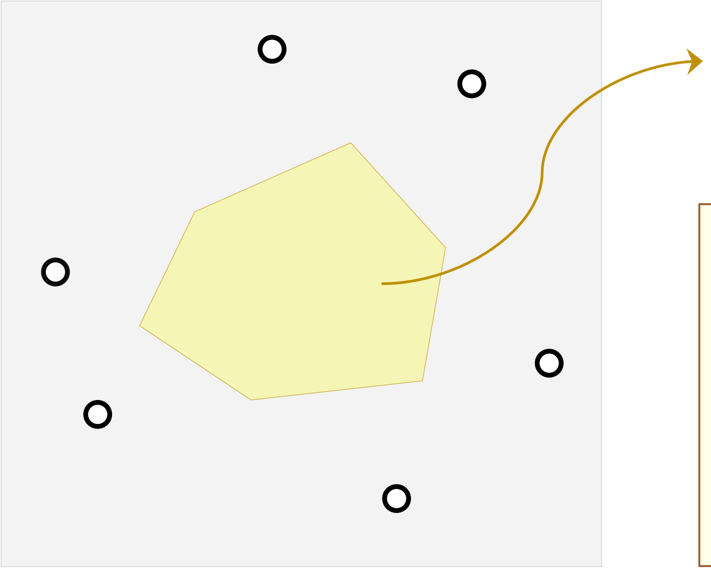


region of **1 – safe points**

N = 6, d = 2, F = 1

Challenges:

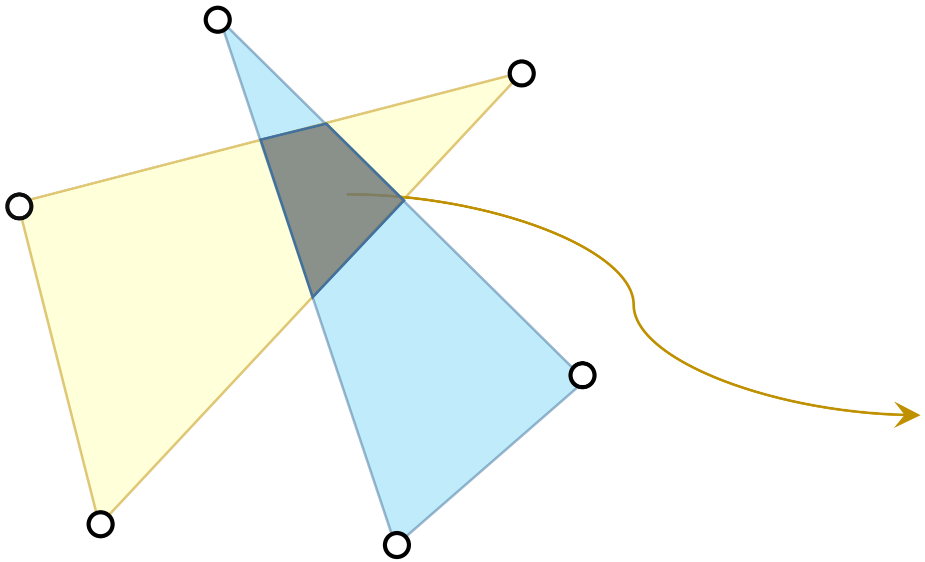
- When can we guarantee ***existence*** of an F – safe point?
- How can we ***find*** it?

Safe Point Using Tverberg Partition (TP)

Park and Hutchinson[1] used **Tverberg partitions (TP)**[2] to compute safe points.

Basic Idea of TP:

Partition points into subsets such that their convex hulls have a non-empty intersection.



Let,
S = no. of subsets in the partition
F = no. of adversary nodes.

$F \leq S - 1$

implies that the intersection contains **F – safe points.**

To compute a safe point, find a point in the intersection.

[1] H. Park and S. Hutchinson, "Fault-tolerant rendezvous of multirobot systems," *IEEE Trans. Robotics, 2017.*

[2] H. Tverberg, "A generalization of Radon's theorem," *J. of the London Math. Society, 1966.*

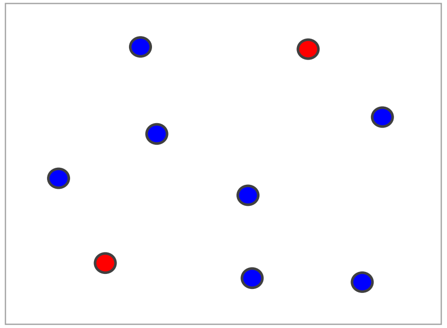
Safe Point Using Tverberg Partition (TP)

Let,

d = dimension of state

N_i = total no. of nodes in the neighborhood of agent i

F_i = no. of adversary agents in the neighborhood of i.



$d = 2$, $N_i = 9$, $F_i = 2$

A **sufficient** condition for the **existence** of an F_i – safe point is

$$F_i \leq \frac{N_i}{d+1} - 1$$

A normal agent can **compute** an F_i – safe point using **TP** in $d^{O(1)} N_i$ time if

$$F_i \leq \left\lceil \frac{N_i}{2^d} \right\rceil - 1$$

$$d \leq 8$$

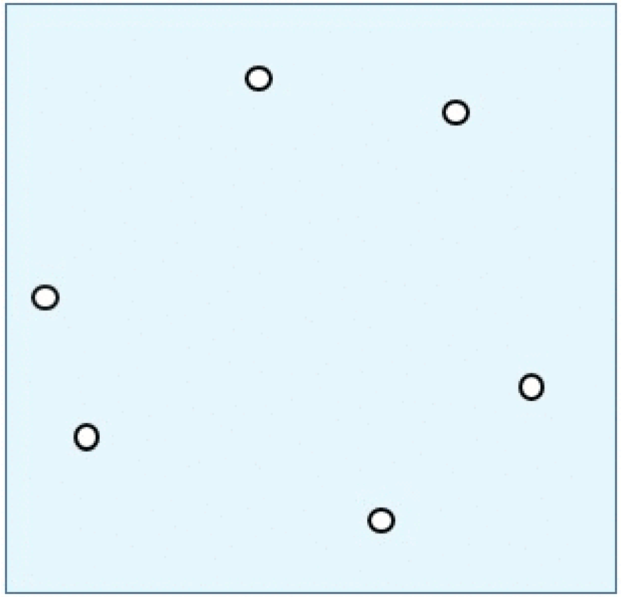
Questions

- **Necessary** condition?
- Can we **improve** the (practical) resilience bound?
- What if ***d* > 8**?

Safe Point Using Centerpoint (CP)

We utilize the notion of centerpoint from discrete geometry.

Centerpoint: For any set S of N points in R^d, a centerpoint c of the set S is a point (not necessarily in S) such that each halfspace containing c contains at least $\frac{N}{d+1}$ points of S.



$N = 6, \quad d = 2$

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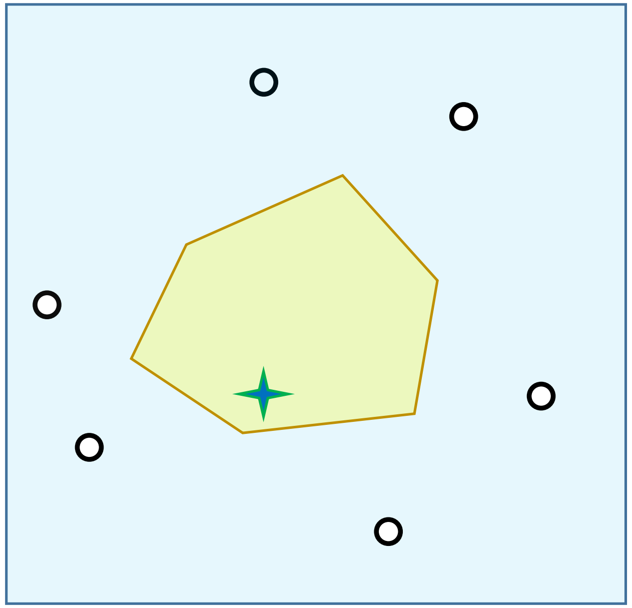


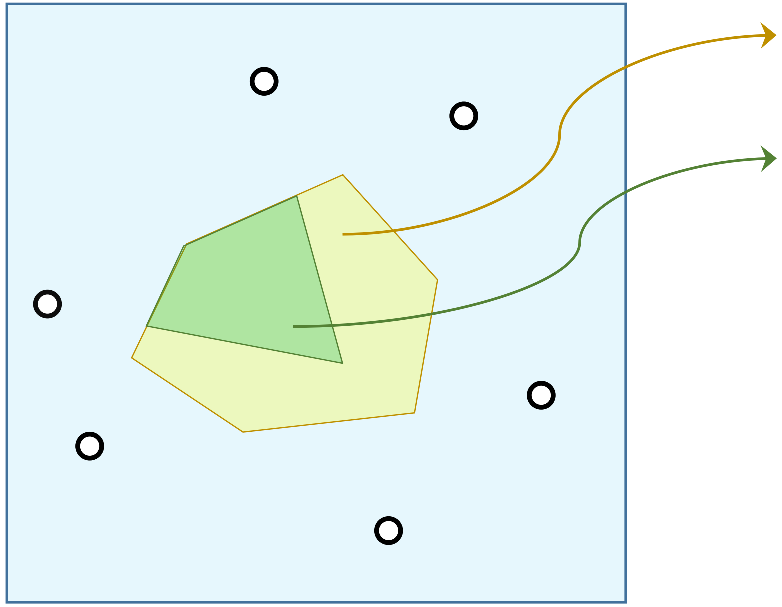
$N = 6, \quad d = 2$

- CP can be viewed as a *generalization of median* in higher dimensions.
- CP *always exists* (CP Theorem).
- CP is *not unique* (CP region).

Safe Point Using Centerpoint (CP)

Theorem: For a set of N points in R^d and $F = \frac{N}{d+1} - 1$, the region of F – safe points is same as the centerpoint region.

(a point is F – safe *if and only if* it is a centerpoint)



1 – safe region ↔ *centerpoint region*

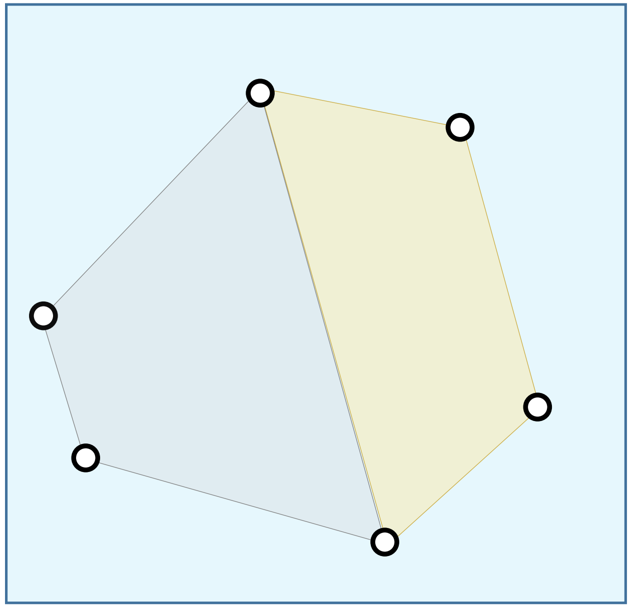
Tverberg points region (not unique)

$N = 6, \quad d = 2, \quad F = 1$

Centerpoint provides a complete characterization of *F*-safe points, whereas Tverberg partitions do not.

Safe Point Using Centerpoint (CP)

For a set of N points in R^d (general positions) and $F \geq \frac{N}{d+1}$, there exist general examples in which an F – safe point does not exist.



$N = 6$, $d = 2$, $F = 2$

There is no 2 – safe point.

(Why? There are two sets with 4 points each such that their convex hulls have an empty intersection.)

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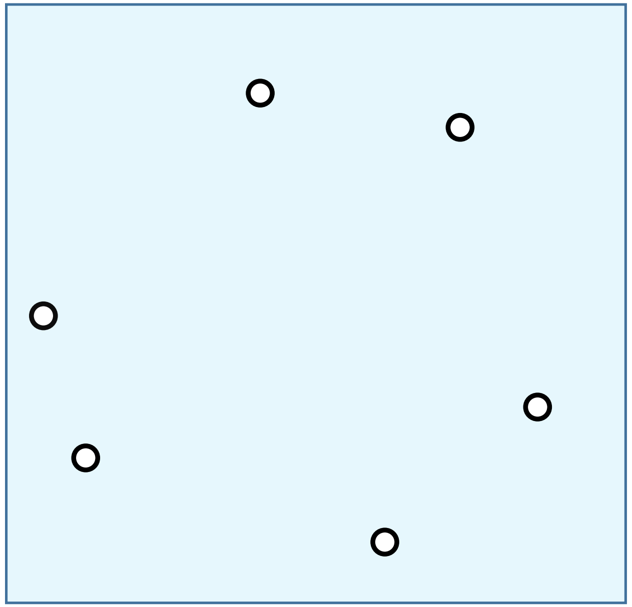


$N = 6, \quad d = 2, \quad F = 2$

There is no 2 – safe point.

(Why? There are two sets with 4 points each such that their convex hulls have an empty intersection.)

A **necessary** condition for the **existence** of an F – safe point is

$$F \leq \frac{N}{d+1} - 1$$

(Previously, we only had a **sufficient** condition.)

Safe Point Computation Using (CP)

Using known results for the centerpoint computation, we can compute an F – safe point if

d = 2 , 3 : $F \leq \frac{N}{d+1} - 1$

d > 3 : $F = \Omega\left(\frac{N}{d^{\frac{r}{r-1}}}\right)$ for any integer r.

Moreover, the time complexity of computing an F – safe point in

- d = 2 is $O(N)$,
- d = 3 is $O(N^2)$, and
- d > 3 is $O\left(N^{c\log d}(rd)^d\right)$ for any integer r.

These bounds are **better** than the ones obtained by using Tverberg partition.

$$F \leq \left\lceil \frac{N}{2^d} \right\rceil - 1$$

ADRC Using Centerpoint

Using centerpoints improve the resilience of ADRC algorithm as compared to Tverberg partition.

(N_i = total no. of agents in the neighborhood of a normal agent i.)

***d* = 2, 3**

CP achieves the *theoretical bound*, that is, ADRC is resilient to $\left(\frac{N_i}{d+1} - 1\right)$ Byzantine adversaries in the neighborhood of agent *i*.

***d* > 3**

Centerpoint: resilient to $\Omega\left(\frac{N_i}{d^2}\right)$ Byzantine adversaries in the neighborhood of *i*.

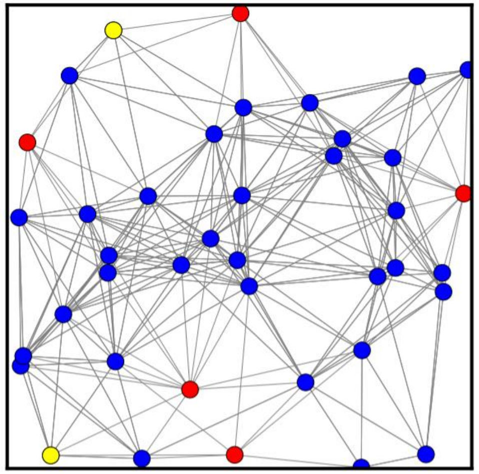
Tverberg: resilient to $\Omega\left(\frac{N_i}{2^d}\right)$ Byzantine adversaries in the neighborhood of *i*.

Simulations

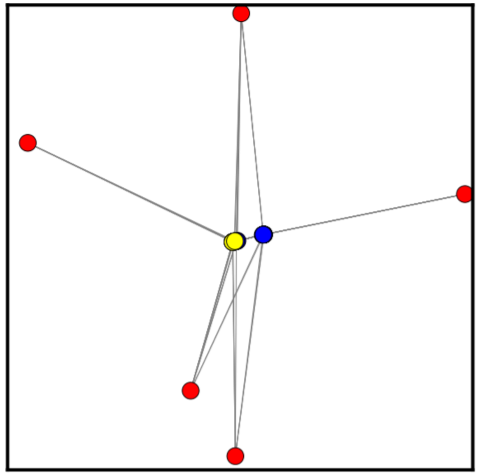
45 robots in a plane (d = 2),

- 5 robots are adversarial (red)
- normal (blue)
- normal robots having $\left(\left\lceil \frac{N_i}{4}\right\rceil - 1\right) < F_i \le \left(\left\lceil \frac{N_i}{3}\right\rceil - 1\right)$ adversaries in their neighborhoods. (yellow)

More adversaries than allowed by the Tverberg-based bound

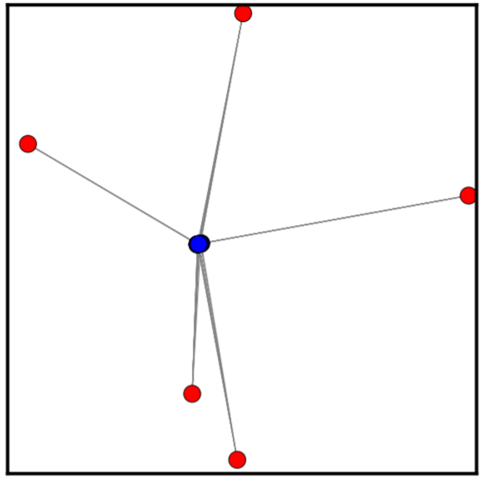


Initial positions



Final positions (**Tverberg** based algo)

No Consensus



Final positions (**Centerpoint** based algo)

Consensus

Conclusions

Resilient vector consensus using **Centerpoint**

Complete characterization of F – safe points

Necessary & sufficient conditions

Improvement in resilience (practically)

Generalization $d > 8$

Extension

Thank You