# Lie's Invariance Condition

*Example* 1

$$\frac{dy}{dx} = y^2 + xy^3 \tag{1}$$

Lie's invariance condition becomes

$$Y_x + (Y_y - X_x)\left(y^2 + xy^3\right) - X_y\left(y^2 + xy^3\right)^2 = y^3\,X + \left(2y + 3xy^2\right)\,Y \tag{2}$$

At this point we will assume a particular form for $X$ and $Y$. We will try to find a solution when we choose

$$X = A(x), \quad Y = B(x)y + C(x) \tag{3}$$

Substituting (3) into (2) and isolating coefficients with respect to $y$ gives the following equations

$$C' = 0, \tag{4a}$$

$$B' - 2C = 0, \tag{4b}$$

$$-A' - B - 3xC = 0, \tag{4c}$$

$$-xA' - A - 2xB = 0. \tag{4d}$$

From (4a) we find that $C = c$, a constant. Substituting into (4b) and solving for $B$ gives

$$B = 2cx + b \tag{5}$$

where $b$ is a second constant of integration. Substituting $B$ and $C$ into the two final equations of (4) gives

$$-A' - 5cx - b = 0, \tag{6a}$$

$$-xA' - A - 4cx^2 - 2bx = 0. \tag{6b}$$

Solving the first for $A$ gives

$$A = -\frac{5}{2}cx^2 - bx + a \tag{7}$$

where $a$ is also constant. Substituting into the final equation in (6) and expanding gives

$$\frac{7}{2}cx^2 - a = 0. \tag{8}$$

Since this must be satisfied for all values of $x$, then we require that $a = 0$ and $c = 0$. Thus, we obtain the infinitesimals

$$X = -bx, \quad Y = by. \tag{9}$$

*Example* 2
Consider

$$\frac{dy}{dx} = \frac{1}{x^2} + \frac{x^2}{xy+1} \tag{10}$$

Lie's invariance condition becomes

$$\begin{aligned} Y_x \quad &+ \quad (Y_y - X_x)\left(\frac{1}{x^2} + \frac{x^2}{xy+1}\right) - X_y\left(\frac{1}{x^2} + \frac{x^2}{xy+1}\right)^2 \\ &= \quad \frac{x^5y + 2x^4 - 2x^2y^2 - 4xy}{x^3(xy+1)^2} X + \frac{x^3}{(xy+1)^2} Y \end{aligned} \tag{11}$$

At this point we will assume a particular form for $X$ and $Y$. We will try to find a solution when we choose

$$X = A(x), \quad Y = B(x)y + C(x) \tag{12}$$

Substituting (12) into (11) and isolating coefficients with respect to $y$ gives the following equations

$$B' = 0, \tag{13a}$$

$$-xA' + 2x^2B' + x^3C' + 2A + xB = 0, \tag{13b}$$

$$-(x^5 + 2x)A' + x^2B' + 2x^3C' - (x^4 - 4)A + 2(x^5 + x)B = 0, \tag{13c}$$

$$-(x + x^5)A' + x^3C' - 2(x^4 - 1)A + (x^5 + x)B + x^6C = 0. \tag{13d}$$

From (13a) we find that $B = b$, a constant. Substituting into (13b) and solving for $C$ gives

$$C = \frac{A}{x^2} + \frac{b}{x} + c \tag{14}$$

where $c$ is a second constant of integration. Substituting $B$ and $C$ into the two final equations of (13) gives

$$xA' + A - 2bx - cx^2 \quad = \quad 0, \tag{15}$$

$$xA' + A - 2bx \quad = \quad 0 \tag{16}$$

which gives $c = 0$ and

$$A = bx + \frac{a}{x}. \tag{17}$$

where $a$ is also constant. Thus, we obtain the infinitesimals

$$X = c_1 x + \frac{c_2}{x}, \quad Y = c_1 y + \frac{2c_1}{x} + \frac{c_2}{x^3} \tag{18}$$

where we have chosen $b = c_1$ and $a = c_2$.

Now we have the infinitesimals, our next job is to reduce the original ODE to one that's separable. As we have a two-parameter family of infinitesimals, we will look at each one separately.

Case 1 $c_1 = 1, c_2 = 0$
In this case $X = x$ and $Y = y + \frac{2}{x}$. Thus, we are require to solve

$$x r_x + \left(y + \frac{2}{x}\right) r_y = 0, \quad x s_x + \left(y + \frac{2}{x}\right) s_y = 1. \tag{19}$$

The solution of each is, respectively

$$r = R\left(\frac{xy+1}{x^2}\right), \quad s = \ln x + S\left(\frac{xy+1}{x^2}\right), \tag{20}$$

where $R$ and $S$ are arbitrary function of their arguments. Here, we will choose simple and choose

$$r = \frac{xy+1}{x^2}, \quad s = \ln x, \tag{21}$$

or

$$x = e^s, \quad y = re^s + e^{-s}. \tag{22}$$

Under this change of variables, (10) becomes

$$\frac{ds}{dr} = -\frac{r}{r^2-1}. \tag{23}$$

This easily integrates giving

$$s = -\frac{1}{2}\ln|r^2-1| + c, \tag{24}$$

and via (21) gives

$$\ln|x| = -\frac{1}{2}\ln|\frac{(xy+1)^2}{x^2} - 1| + c, \tag{25}$$

or, after some simplification

$$\frac{(xy+1)^2}{x^2} - x^2 = c, \tag{26}$$

the exact solution of (10).

Case 2 $c_1 = 0, c_2 = 1$
In this case $X = \dfrac{1}{x}$ and $Y = \dfrac{1}{x^3}$. Thus, we are require to solve

$$\frac{1}{x}\,r_x + \frac{1}{x^3}\,r_y = 0, \quad \frac{1}{x}\,s_x + \frac{1}{x^3}\,s_y = 1. \tag{27}$$

The solution of each is, respectively

$$r = R\left(\frac{xy+1}{x}\right), \quad s = \frac{1}{2}x^2 + S\left(\frac{xy+1}{x}\right), \tag{28}$$

where $R$ and $S$ are arbitrary function of their arguments. Here, we will choose simple and choose

$$r = \frac{xy+1}{x}, \quad s = \frac{1}{2}x^2, \tag{29}$$

or

$$x = \sqrt{2s}, \quad y = r - \frac{1}{\sqrt{2s}}. \tag{30}$$

Under this change of variables, (10) becomes

$$\frac{ds}{dr} = r. \tag{31}$$

This easily integrates giving

$$s = \frac{1}{2}r^2 + c, \tag{32}$$

and via (29) gives exactly (26).