Complex Numbers

⇒ Given $z_1 = 5 + i$ and $z_2 = -2 + 3i$.

(i) Find $(z_1)^2$

(ii) Show that $|z_1|^2 = 2|z_2|^2$

$|a+bi| = \sqrt{a^2+b^2}$

$(5+i)^2$

$(5+i)(5+i)$

$25 + 5i + 5i + i^2$

$25 + 10i + (-1)$

$24 + 10i$

$|5+i| = \sqrt{(5)^2+(1)^2} = \sqrt{26}$

$|z_2| = |-2+3i| = \sqrt{(-2)^2+(3)^2}$

$= \sqrt{13}$

So: $2|z_2|^2 = 2(\sqrt{13})^2 = 2(13)$

26

Altogether:

$(\sqrt{26})^2 = 26$ ✓

⇒ Solve the equation $z^2 + 4z + 20 = 0$, giving your answer in the form $a + bi$.

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$a=1 \quad b=4 \quad c=20$

$$z = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(20)}}{2(1)}$$

$$z = \frac{-4 \pm \sqrt{-4}}{2}$$

$\sqrt{-4} = \sqrt{4}\sqrt{-1}$

$2i$

$$z = \frac{-4 \pm 2i}{2} = -\frac{4}{2} \pm \frac{2i}{2}$$

$$\boxed{-2 \pm i}$$

⇒ Write $\frac{5-5i}{2+i}$ in the form $a+bi, a, b \in R$.

$z(2+i) = 5-5i$ find z

$$z = \frac{(5-5i)}{(2+i)} \times \frac{(2-i)}{(2-i)}$$

$$z = \frac{10-5i-10i+5i^2}{4-\cancel{2i}+\cancel{2i}-i^2} = \frac{10-15i+5(-1)}{4-(-1)}$$

$$z = \frac{10-15i-5}{4+1} = \frac{5-15i}{5} = \frac{5}{5} - \frac{15}{5}i$$

$$1-3i$$

⇒ Let z be the complex number $-1+\sqrt{3}i$.

(i) Express z^2 in the form $a+bi$.

(ii) Find the value of the real number p such that z^2+pz is real.

$0i$

$z^2 = (-1+\sqrt{3}i)^2$

$z^2 = (-1+\sqrt{3}i)(-1+\sqrt{3}i)$

$1-\sqrt{3}i-\sqrt{3}i+3i^2$

$1-2\sqrt{3}i+3(-1)$

$1-2\sqrt{3}i-3$

(i) $-2-2\sqrt{3}i$

(ii) $(-2-2\sqrt{3}i)+p(-1+\sqrt{3}i)$ is real

$-2\boxed{-2\sqrt{3}i}-p\boxed{+p\sqrt{3}i}$

$-2\sqrt{3}i+p\sqrt{3}i = 0i$

$-2\sqrt{3}+p\sqrt{3} = 0$

$-2\sqrt{3} = -p\sqrt{3}$

$$\frac{2\sqrt{3}}{\sqrt{3}} = \frac{p\sqrt{3}}{\sqrt{3}}$$

$2 = p$

$\Rightarrow$ Given that $2 + 3i$ is a root of $2z^3 - 9z^2 + 30z - 13 = 0$, find the other two roots.

$2-3i$ is also a root

Any quadratic is of the form:

$$z^2 - (\text{sum of roots})z + (\text{product of roots}) = 0$$

$$z^2 - \big((2+3i)+(2-3i)\big)z + \big((2+3i)(2-3i)\big) = 0$$

$$z^2 - (4)z + (4 \cancel{-6i} \cancel{+6i} - 9i^2) = 0$$

$$z^2 - 4z + (4 - 9(-1)) = 0$$

$$z^2 - 4z + (4+9) = 0$$

$z^2 - 4z + 13 \longrightarrow$ is quadratic factor!

Now divide in!

$$\begin{array}{r l}
 & 2z - 1 \\
z^2 - 4z + 13 & \overline{)\, \cancel{2z^3} - 9z^2 + 30z - 13} \\
 & \ominus \cancel{2z^3} \oplus 8z^2 \mp 26z \quad \downarrow \\
\hline
 & \quad -\cancel{z^2} + 4z - 13 \\
 & \quad \oplus \cancel{-z^2} \oplus 4z \ominus 13 \\
\hline
 & \qquad 0
\end{array}$$

$2z - 1 = 0$

$2z = 1$

$z = \frac{1}{2}$

$\boxed{2+3i}$

$\boxed{2-3i}$

$\boxed{\frac{1}{2}}$

⇒ Simplify the following expression giving your answer in the form $a + bi, a, b \in \mathbb{R}$:

$$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^2 \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^4$$

$$\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)$$

$$1\left(\cos\left(\frac{2\pi}{3} + \frac{8\pi}{3}\right)\right) + i\sin\left(\frac{2\pi}{3} + \frac{8\pi}{3}\right)$$

$$\cos\left(\frac{10\pi}{3}\right) + i\sin\left(\frac{10\pi}{3}\right)$$

$$-\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

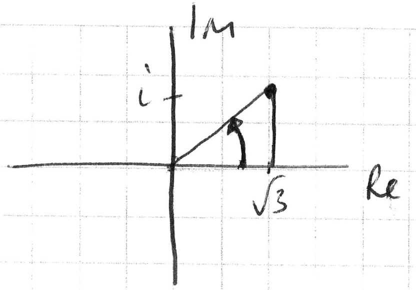
⇒ Express $\sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$.
Use de Moivre's theorem to simplify $\boxed{(\sqrt{3} + i)^{11}}$.

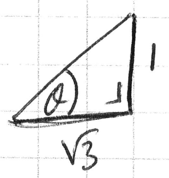
$r = \sqrt{a^2 + b^2}$

$r = \sqrt{(\sqrt{3})^2 + (1)^2}$

$r = \sqrt{3+1}$

$r = 2$





$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta = 30°$

$2(\cos 30° + i\sin 30°)$

$\times\frac{\pi}{180}$ $\times\frac{\pi}{180}$

or

$2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$2^{11}(\cos(11\times 30°) + i\sin(11\times 30°))$

$2048(\cos(330°) + i\sin(330°))$

$2048\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$

$2048\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

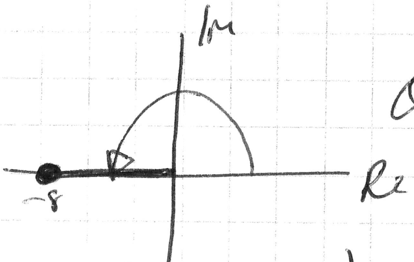
Find the values of z for which $z^3 = -8$, giving your answer in $a + bi$ form.

$$z = (-8+0i)^{1/3} \qquad (+360^\circ n \text{ to } \theta)$$

$$r = \sqrt{(-8)^2 + (0)^2}$$

$$r = \sqrt{64}$$

$$r = 8$$



$$\theta = 180^\circ$$

$$8\left(\cos(180^\circ + 360^\circ n) + i\sin(180^\circ + 360^\circ n)\right)$$

$$8^{\frac{1}{3}}\left(\cos\left(\tfrac{1}{3}(180^\circ + 360^\circ n)\right) + i\sin\left(\tfrac{1}{3}(180^\circ + 360^\circ n)\right)\right)$$

$$2\left(\cos(60^\circ + 120^\circ n) + i\sin(60^\circ + 120^\circ n)\right)$$

$n=0 \quad 2(\cos 60^\circ + i\sin 60^\circ) = 2(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i) = 1 + \sqrt{3}i$

$n=1 \quad 2(\cos(60^\circ + 120^\circ) + i\sin(60^\circ + 120^\circ)) = 2(-1 + 0i) = -2 + 0i$

$n=2 \quad 2(\cos(60^\circ + 240^\circ) + i\sin(60^\circ + 240^\circ)) = 2(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i) = 1 - \sqrt{3}i$

Given that $\bar{z}$ is the conjugate of z and $z = a + bi$ where $\boxed{a \text{ and } b \text{ are real}}$, find the possible values of z if $z\bar{z} - 2iz = 7 - 4i$.

$$\bar{z} = a - bi$$

$$(a+bi)(a-bi) - 2i(a+bi) = 7 - 4i$$

$$a^2 - abi + abi - b^2i^2 - 2ai - 2bi^2 = 7 - 4i$$

$$a^2 - b^2(-1) - 2ai - 2b(-1) = 7 - 4i$$

$$a^2 + b^2 \boxed{-2ai} + 2b = 7 \boxed{-4i}$$

$$a^2 + b^2 + 2b = 7 \qquad -2ai = -4i$$

$$-2a = -4$$

$$a = 2$$

$$(2)^2 + b^2 + 2b = 7$$

$$4 + b^2 + 2b = 7$$

$$b^2 + 2b - 3 = 0$$

$$(b-1)(b+3) = 0 \rightarrow \text{So } b = 1 \text{ or } b = -3$$

$$z = 2 + i \text{ or } z = 2 - 3i$$

⇒ Simplify $4i^{13} + 3i^3$.

$i^2 = -1$

$i^3 = (i^2)i$

$= (-1)i$

$= -i$

$* \; i^4 = (i^2)(i^2)$

$= (-1)(-1)$

$= 1$

$4(i^4)^3(i^1) + 3i^3$

$4(1)^3(i) + 3(-i)$

$4i - 3i$

i

Algebra

Given $f(x) = 2x^3 + 13x^2 + 13x - 10$.

Show that $f(-2) = 0$ and hence find the three factors of $f(x)$.

$2(-2)^3 + 13(-2)^2 + 13(-2) - 10 = 0$

$-16 + 52 - 26 - 10 = 0$

$0 = 0$

$f(-2) = 0$ then $x+2$ is a factor.

$$\begin{array}{r|l} & 2x^2 + 9x - 5 \\ \hline x+2 & 2x^3 + 13x^2 + 13x - 10 \\ & \ominus 2x^3 \mp 4x^2 \\ \hline & \quad 9x^2 + 13x \\ & \ominus 9x^2 \mp 18x \\ \hline & \quad -5x - 10 \\ & \oplus -5x \oplus -10 \\ \hline & \quad 0 \end{array}$$

$2x^2 + 9x - 5$ factorise!

$2x \cdot -5 = -10$ → $+10, -1$

$2x^2 + 10x - x - 5$

$2x(x+5) - 1(x+5)$

$(2x-1)(x+5)$

$(x+2), (2x-1), (x+5)$ factors

Roots: $x = -2$, $2x - 1 = 0$, $x = -5$

$2x = 1$

$x = \frac{1}{2}$

Solve each of the following equations

(i) $x^3 - 4x^2 - x + 4 = 0$

Try $x = 1$, $(1)^3 - 4(1)^2 - (1) + 4 = 0$

$(x-1)$ is a factor!

$$\begin{array}{r|l} & x^2 - 3x - 4 \\ \hline x-1 & x^3 - 4x^2 - x + 4 \\ & \ominus x^3 \oplus x^2 \\ \hline & \quad -3x^2 - x \\ & \oplus 3x^2 \mp 3x \\ \hline & \quad -4x + 4 \\ & \quad -4x + 4 \\ \hline \end{array}$$

$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

$x = -1$, $x = 4$

$x = 1$, $x = -1$, $x = 4$

$(x-1), (x+1), (x-4)$

Prove that for all real numbers p and q,

(i) $(p+q)^2 \leqslant 2(p^2+q^2)$.

$$(p+q)(p+q) \leq 2p^2 + 2q^2$$

$$p^2 + pq + pq + q^2 - 2p^2 - 2q^2 \leq 0$$

$$-p^2 + 2pq - q^2 \leq 0$$

*multiply across by -1 (change all signs, flip inequality)

$$p^2 - 2pq + q^2 \geqslant 0$$

$$(p-q)^2 \geqslant 0.$$

Solve each of these equations and check each solution:

$\sqrt{x+7} + \sqrt{x} = 7$

$$\sqrt{x+7} = 7 - \sqrt{x}$$

$$\left(\sqrt{x+7}\right)^2 = (7 - \sqrt{x})^2$$

$$x + 7 = (7 - \sqrt{x})(7 - \sqrt{x})$$

$$x + 7 = 49 - 7\sqrt{x} - 7\sqrt{x} + x$$

$$\cancel{x} + 7 = 49 - 14\sqrt{x} + x$$

$$-\cancel{x} - 49 \quad -\cancel{49} \quad -x$$

$$\frac{-42}{-14} = \frac{-\cancel{14}\sqrt{x}}{-\cancel{14}}$$

$$3 = \sqrt{x}$$

$$(3)^2 = (\sqrt{x})^2$$

$$9 = x$$

*verify your answers!

$$\sqrt{9+7} + \sqrt{9} = 7$$

$$\sqrt{16} + 3 = 7$$

$$4 + 3 = 7 \checkmark$$

For what value(s) of k does each of the following equations have equal roots?

(i) $4x^2 + kx + 9 = 0$

$b^2 - 4ac = 0$

$a = 4 \quad b = k \quad c = 9$

$(k)^2 - 4(4)(9) = 0$

$k^2 - 144 = 0$

$(k+12)(k-12) = 0$

$k = -12$ or $k = 12$.

$b^2 - 4ac = 0$

Real Roots (2 different)

$\to b^2 - 4ac > 0$

Unreal Roots / imaginary

$\to b^2 - 4ac < 0$

Given that (any real number)$^2 \geqslant 0$, prove that the following equations have real roots for all values of $k \in R$.

(i) $kx^2 + 2x + (2 - k) = 0$

$b^2 - 4ac \geqslant 0$

$a = k \quad b = 2 \quad c = 2 - k$

$(2)^2 - 4(k)(2-k) \geqslant 0$

$4 - 4k(2-k) \geqslant 0$

$4 - 8k + 4k^2 \geqslant 0$

$4k^2 - 8k + 4 \geqslant 0$

$k^2 - 2k + 1 \geqslant 0$

$(k-1)^2 \geqslant 0$

Aside.

$(k-1)^2$

$k^2 - 2k + 1$

Simplify each of the following.

(i) $\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1}$

$$\times(\sqrt{2}+1)\quad \frac{1(\sqrt{2}+1) - 1(\sqrt{2}-1)}{(\sqrt{2}-1)(\sqrt{2}+1)} \quad \times(\sqrt{2}-1)$$

$$\frac{\cancel{\sqrt{2}}+1-\cancel{\sqrt{2}}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} \longrightarrow \frac{2}{(\sqrt{2}-1)(\sqrt{2}+1)}$$

Show that $(2\sqrt{5} - 3\sqrt{2})(2\sqrt{5} + 3\sqrt{2}) = 2$.

$$\sqrt{5}\times\sqrt{5} = \sqrt{5\times5}$$
$$= \sqrt{25}$$
$$= 5$$

$4\sqrt{25} \qquad -9\sqrt{4}$

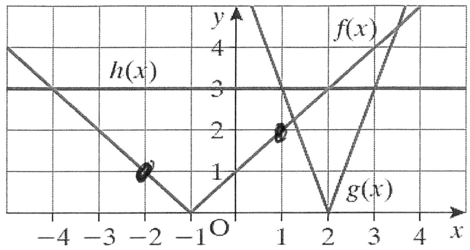
$4(5) + \cancel{6\sqrt{10}} - \cancel{6\sqrt{10}} - 9(2)$

$4(5) - 9(2)$

$20 - 18$

2.

Use the graphs of the functions $f(x) = |x + 1|$, $g(x) = |3x - 6|$ and $h(x) = 3$ to estimate the range of values of x that satisfies each of the following inequalities.



$f(x) = |x+1|$

$y = |x+1|$

$x = 1, \quad y = |1+1| = 2 \quad (1, 2)$

$x = -2, \quad y = |-2+1| = y = |-1| = 1 \quad (-2, 1)$

Solve $2x^2 - \sqrt{3}x - 3 = 0$.

$a = 2 \quad b = -\sqrt{3} \quad c = -3$

$$x = \frac{-(-\sqrt{3}) \pm \sqrt{(-\sqrt{3})^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{\sqrt{3} \pm \sqrt{27}}{4}$$

$x = \sqrt{3}$ or $x = -\frac{\sqrt{3}}{2}$

Solve each of the following inequalities for $x \in R$.

(i) $|2x - 1| = 11$ (ii) $|3x + 5| = 4|x-2|$

$2x-1=11$ or $2x-1=-11$

$2x=12$ or $2x=-10$

$x=6$ $x=-5$

$(|3x+5|)^2 = (4|x-2|)^2$

$(3x+5)(3x+5) = 16(x-2)(x-2)$

$9x^2+15x+15x+25 = 16(x^2-2x-2x+4)$

$9x^2+30x+25 = 16(x^2-4x+4)$

$9x^2+30x+25 = 16x^2-64x+64$

$0 = 7x^2-94x+39$

$$x = \frac{-(-94) \pm \sqrt{(-94)^2 - 4(7)(39)}}{2(7)}$$

$x = 13$ or $x = 3/7$

Find the range of values of x for which

$$\frac{3x+4}{x-5} > 2, x \neq 5$$

$$(x-5)^2\left(\frac{3x+4}{\cancel{x-5}}\right) > 2(x-5)^2$$

$(x-5)(3x+4) > 2(x-5)(x-5)$

$3x^2+4x-15x-20 > 2[x^2-5x-5x+25]$

$3x^2-11x-20 > 2(x^2-10x+25)$

$3x^2-11x-20 > 2x^2-20x+50$

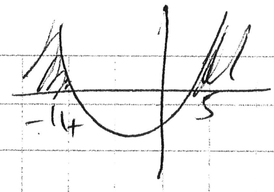
$x^2+9x-70 > 0$ * Quadratic Inequality

Consider $x^2+9x-70=0$

$(x-5)(x+14)=0$

$x=5$ $x=-14$

Sketch or test in between roots



$x<-14$, $x>5$

or Test in between roots?

try $x=0$

$x^2+9x-70>0$

$(0)^2+9(0)-70>0$

$-70>0$ No!

So beyond roots

$\underline{\underline{x<-14, x>5}}$.

Solve for x and y:

$$3x + y = 7$$
$$x^2 + y^2 = 13$$

$y = 7 - 3x$

$x^2 + (7-3x)^2 = 13$

$x^2 + 49 - 42x + 9x^2 = 13$

$10x^2 - 42x + 36 = 0$

$$x = \frac{-(-42) \pm \sqrt{(-42)^2 - 4(10)(36)}}{2(10)}$$

$x = 3$ or $x = \frac{6}{5}$

$x = 3, \quad y = 7 - 3x$

$y = 7 - 3(3) = -2$

$x = \frac{6}{5}, \quad y = 7 - 3\left(\frac{6}{5}\right) = \frac{17}{5}$

$x = 3, \ y = -2$

$x = \frac{6}{5}, \ y = \frac{17}{5}$.

Solve the simultaneous equations

$$3x + 5y - z = -3$$
$$2x + y - 3z = -9$$
$$x + 3y + 2z = 7.$$

$3x + 5y - z = -3 \quad (\times -3)$

$2x + y - 3z = -9$

$-9x - 15y + 3z = 9$

$2x + y - 3z = -9$

$\boxed{-7x - 14y = 0}$

$3x + 5y - z = -3 \quad (\times 2)$

$x + 3y + 2z = 7$

$6x + 10y - \cancel{2z} = -6$

$x + 3y + \cancel{2z} = 7$

$\boxed{7x + 13y = 1}$

$-\cancel{7x} - 14y = 0$

$\cancel{7x} + 13y = 1$

$-y = 1$

$y = -1$

$y = -1, \quad 7x + 13y = 1$

$7x + 13(-1) = 1$

$7x = 1 + 13$

$7x = 14$

$x = 2$

$x = 2, \ y = -1, \quad 3x + 5y - z = -3$

$3(2) + 5(-1) - z = -3$

$6 - 5 - z = -3$

$-z = -3 - 1$

$-z = -4$

$z = 4$

Simplify each of the following algebraic expressions.

$$\frac{2+\frac{x}{2}}{x^2-16}$$

$\otimes$ $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$

$\div$ $\frac{2}{3} \div \frac{1}{5} \rightarrow \frac{2}{3} \times \frac{5}{1} = \frac{10}{3}$

$$\frac{2+\frac{x}{2}}{(x+4)(x-4)}$$

$$\frac{2(2)+\cancel{2}\left(\frac{x}{\cancel{2}}\right)}{2(x+4)(x-4)}$$

$$\frac{1\cancel{(4+x)}}{2\cancel{(x+4)}(x-4)} \rightarrow \frac{1}{2(x-4)}.$$

Solve the following equations:

(iii) $\frac{2}{x-2}+\frac{3}{x}=\frac{5}{x-4}$

$$(x)\cancel{(x-2)}(x-4)\left(\frac{2}{\cancel{x-2}}\right)+\cancel{(x)}(x-2)(x-4)\left(\frac{3}{\cancel{x}}\right)=(x)(x-2)\cancel{(x-4)}\left(\frac{5}{\cancel{x-4}}\right)$$

$$2x(x-4)+3(x-2)(x-4)=5x(x-2)$$

$$2x^2-8x+(3x-6)(x-4)=5x^2-10x$$

$$\boxed{2x^2}\,-8x\,\boxed{+3x^2}-12x-6x+24=5x^2-10x$$

$$\begin{array}{rcl} \cancel{5x^2}-26x+24 &=& \cancel{5x^2}-10x \\ -5x^2 & & -5x^2 \end{array}$$

$$\begin{array}{rcl} \cancel{-26x}+24 &=& -10x \\ +26x & & +26x \end{array}$$

$$24=16x$$

$$\frac{3}{2}=\frac{24}{16}=x$$

Simplify the following:

$$\frac{1}{x^2-9} - \frac{2}{x^2-x-6}$$

$$x(x+2)\left(\frac{1}{(x+3)(x-3)} - \frac{2}{(x+2)(x-3)}\right) x(x+3)$$

$$\frac{1(x+2) - 2(x+3)}{(x+3)(x-3)(x+2)}$$

$$\frac{x+2-2x-6}{(x+3)(x-3)(x+2)} \rightarrow \frac{-x-4}{(x+3)(x-3)(x+2)}$$

Sequences & Series

In a geometric series, the first term is 12 and the sum to infinity is 36.
Find the common ratio.

$T_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r}$ $\quad S_\infty = \frac{a}{1-r}$

$a = 12$

$S_\infty = \frac{12}{1-r}$

$36 = \frac{12}{1-r}$

$(1-r)(36) = \left(\frac{12}{\cancel{1-r}}\right)(\cancel{1-r})$

$36 - 36r = 12$

$\cancel{36} - 36r = 12$
$-36 \qquad\qquad -36$

$\frac{\cancel{-36}r}{\cancel{-36}} = \frac{-24}{-36}$

$r = \frac{2}{3}$

Evaluate $\sum_{r=3}^{16} (2r + 1)$. Sum!

$(2(3)+1) + (2(4)+1) + (2(5)+1) + (2(6)+1) + \ldots + (2(16)+1)$

$\underbrace{7 + 9}_{+2} \; \underbrace{+ 11}_{+2} \; \underbrace{+ 13}_{+2} + \ldots + 33$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{14} = \frac{14}{2}[2(7) + (14-1)(2)]$

$S_{14} = 280$

The fifth term of an arithmetic sequence is twice the second term.
The two terms also differ by 9.
Find the sum of the first 10 terms of the sequence.

$T_n = a + (n-1)d$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$T_5 = 2 \times T_2$

$a + 4d = 2(a + d)$

$a + 4d = 2a + 2d$

$0 = a - 2d$

$0 = a - 2(3)$

$0 = a - 6$

$6 = a$

$T_5 - T_2 = 9$

$(a + 4d) - (a + d) = 9$

$a + 4d - a - d = 9$

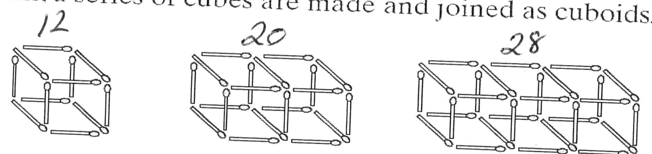
$3d = 9$

$d = 3$

$S_{10} = \frac{10}{2}[2(6) + (10-1)(3)]$

$S_{10} = 5[39] = 195$

Using matchsticks, a series of cubes are made and joined as cuboids, as shown in the diagram.



(i) Determine the number of matchsticks needed for the nth cuboid.
(ii) Determine the maximum number of cubes in the cuboid if there are 2006 matchsticks left for the construction.

(i) 12, 20, 28

$\underbrace{12, 20}_{+8}, \underbrace{20, 28}_{+8}$

arithmetic!

So either $T_n = a + (n-1)d$

or $S_n = \frac{n}{2}[2a + (n-1)d]$

$T_n = 12 + (n-1)(8)$

$T_n = 12 + 8n - 8$

$T_n = 4 + 8n$

or

$T_n = 8n + 4$

(ii) Which term?

$2006 = 8n + 4$

$-4 \quad -4$

$\frac{2002}{8} = \frac{8n}{8}$

$250.25 = n$

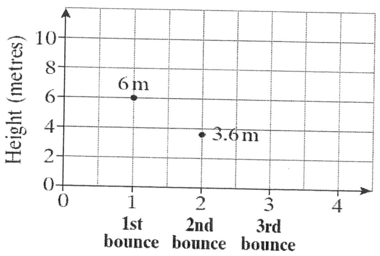
250th term $\therefore$ 250 cubes!

A ball is dropped from a height of 10 m and bounces back up to 6m and then to 3.6 m and so on, as shown in the diagram. Copy the graph and add in the next five heights of the ball from the ground.

(i) Write down a series of numbers representing the total distance travelled by the ball.

(ii) Describe what type of series is generated. Geometric

(iii) Find the total distance travelled by the ball.



10 6 3.6

$\times \frac{3}{5}$ $\times \frac{3}{5}$

Geometric

(i) 10, 6, 6, 3.6, 3.6, 2.16, 2.16, 1.296, 1.296, 0.7776, 0.7776

$\underbrace{6 + 3.6 + 2.16 + 1.296 + \ldots}$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_\infty = \frac{6}{1-\left(\frac{2}{5}\right)} = 15$

$S_\infty = \frac{a}{1-r}$

$10 + 2(15) = \underline{40m}$

Financial Maths

Kris is investing in an annuity by saving €3000 per year at 7.3% p.a. for 8 years.

(i) Find the final value of the annuity if the investment is made at the start of each year.

(ii) By finding the present value of all his payments, calculate the single amount of money which could be invested at the same rate and for the same amount of time to give the same final payment.

(iii) Find the final value of the sum of money from (ii) if it is invested at 7.3% for 8 years.

(i) $3000(1+0.073)^1 + 3000(1+0.073)^2 + \dots + 3000(1+0.073)^8$

$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$ $a = 3000(1.073)$ $r = 1.073$ $n = 8$

$$\frac{3000(1.073)(1-(1.073)^8)}{1-(1.073)} = €33{,}385.23$$

(ii) $P = \frac{F}{(1+i)^t}$ so: $P = \frac{33385.23}{(1+0.073)^8} = €19000.13$

(iii) $F = P(1+i)^t$ so: $F = 19000.13(1+0.073)^8 = €33{,}385.23$

John makes savings of €200 at the beginning of each month for 5 months at an effective monthly rate of 0.75%.
Express these savings as a geometric series.
Write down the first term, the common ratio and an expression for the sum of the five terms.

$F = P(1+i)^t$

$200(1+0.0075)^1 + 200(1+0.0075)^2 + \dots + 200(1+0.0075)^5$

$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$ $a = 200(1.0075)$ $r = 1.0075$ $n = 5$

$$\frac{200(1.0075)(1-(1.0075)^5)}{1-(1.0075)}$$

A bank offers you a rate of 10% on a 20-year mortgage to be paid in monthly repayments. $\rightarrow 20 \times 12 = 240$ MTHS.
If the most you can afford to pay in monthly repayments is €700, find the value of the biggest mortgage you can afford.

$$(1+r)^{12} = 1 + 0.10$$

$$1+r = \sqrt[12]{1.1}$$

$$r = \sqrt[12]{1.1} - 1$$

$$r = 0.007974$$

$$A = \frac{Pi(1+i)^t}{(1+i)^t - 1}$$

$$700 = \frac{P(0.007974)(1+0.007974)^{240}}{(1+0.007974)^{240} - 1}$$

$$700 = P(0.009366285159)$$

$$€74736.14 = P$$

Your company has an expected pension liability of €500 000 in 10 years time.

(i) What amount of money would you now require to cover this expected liability. Assume an annual rate of 9%.

(ii) How much would you need to set aside at the end of each year for the next 10 years to cover the liability (assume the same rate applies)?

(i) $P = \frac{F}{(1+i)^t}$

$$P = \frac{500000}{(1+0.09)^{10}}$$

$$P = €211,205.40$$

(ii) $F = P(1+i)^t$

$$P + P(1+0.09) + P(1+0.09)^2 + \ldots + P(1+0.09)^9$$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}} \quad a = P \quad r = 1.09 \quad n = 10$$

$$\frac{P(1-(1.09)^{10})}{1-(1.09)}$$

$$500,000 = \frac{P(1-(1.09)^{10})}{1-(1.09)}$$

$$500,000 = P(15.19292972)$$

$$€32,910.04 = P$$

Silvia is planning an overseas trip lasting 3 years and she estimates that she will need €600 per month for expenses. $\longrightarrow 3 \times 12 = 36$ mths
How much money does she need to have saved to fund this trip?
Assume an average rate of interest of 4% over the period of the trip.

$(1+r)^{12} = 1+0{,}04$

$1+r = \sqrt[12]{1{,}04}$

$r = \sqrt[12]{1{,}04} - 1$

$r = 0.003274$

$A = P\dfrac{i(1+i)^t}{(1+i)^t - 1}$

$600 = P\dfrac{(0.003274)(1+0.003274)^{36}}{(1+0.003274)^{36} - 1}$

$600 = P(0.02949232252)$

$€20344.28 = P$

Logs & Indices

Solve the equation $\ln(x - 1) + \ln(x + 2) = \ln(6x - 8)$.

$\log_{\square} \square = \square \rightarrow$ power form

OR

$\log_{\square} \square = \log_{\square} \square$

E.g. $\log_2 -4 = x$

$2^x = -4$

$$\ln\boxed{(x-1)(x+2)} = \ln\boxed{(6x-8)}$$

$$(x-1)(x+2) = 6x-8$$

$$x^2+2x-x-2 = 6x-8$$

$$x^2-5x+6=0$$

$$(x-2)(x-3)=0$$

$x=2 \quad x=3 \longrightarrow$ check validity of both answers.

$y = Ae^{bt}$. Given that $y = 6$ when $t = 1$, and $y = 8$ when $t = 2$, find the values of A and b.

$y = Ae^{bt}$ and $y = Ae^{bt}$

$6 = Ae^{b(1)}$ $\qquad$ $8 = Ae^{b(2)}$

$6 = Ae^{b}$ $\qquad$ $8 = Ae^{2b}$

$\frac{6}{A} = e^{b}$ $\qquad$ $\frac{8}{A} = e^{2b}$

$\frac{8}{A} = (e^b)^2$

$\frac{8}{A} = \left(\frac{6}{A}\right)^2$

$\rightarrow \frac{8}{A} = \frac{36}{A^2}$

$$(A^2)(A)\left(\frac{8}{\cancel{A}}\right) = (\cancel{A^2})(A)\left(\frac{36}{\cancel{A^2}}\right)$$

$$8A^2 = 36A$$

$$8A^2 - 36A = 0$$

$$4A(2A-9) = 0$$

$4A = 0 \qquad 2A = 9$

$\underline{A=0} \qquad \underline{\underline{A = \frac{9}{2}}}$

when $A = \frac{9}{2}$, $\frac{6}{\left(\frac{9}{2}\right)} = e^b$, $\log_e\left(\frac{4}{3}\right) = b$

$0.2877 = b$

when $A = 0$, $\frac{6}{0} = e^b$ ERROR!

An exponential function is defined by $f(x) = 3 \times 4^x$. Find

(i) the value of a if $(a, 6)$ lies on $f(x)$

(ii) the value of b if $\left(\frac{-1}{2}, b\right)$ lies on $f(x)$.

(i) $f(x) = 3 \times 4^x$

or $y = 3 \times 4^x$

$(a, 6)$
x, y

$6 = 3 \times 4^a$

$\frac{6}{3} = \frac{\cancel{3} \times 4^a}{\cancel{3}}$

$2 = 4^a$

$\log_4 2 = a$

$\frac{1}{2} = a$

(ii) $\left(-\frac{1}{2}, b\right)$
$x \quad y$

$y = 3 \times 4^x$

$b = 3 \times 4^{(-\frac{1}{2})}$

$b = \frac{3}{2}$

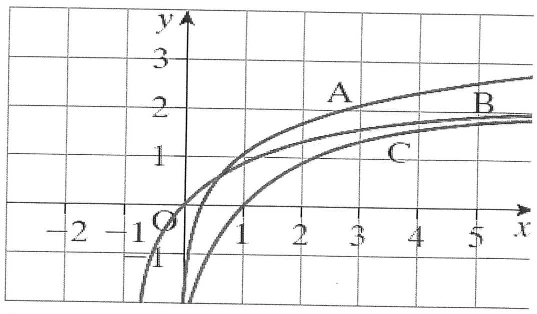
The graphs of the log functions

(i) $\ln(x)$ C

(ii) $\ln(x + 1)$ B

(iii) $\ln(x) + 1$ A

are shown in this diagram. Identify each curve, giving a reason for your answers.



→ goes through pt. (1,0)

→ ln(x+1) horizontal translation 1 left

→ ln(x) +1 vertical translation 1 up

The mass M of a radioactive material remaining after t years is given by the formula

$$M = 30 \times 2^{-0.001t} \text{ grams. Find}$$

(i) the original mass

(ii) how long it would take for the material to decay to 10 grams

(iii) how long it would take to decay to the "safe level" of 1% of its original mass.

(i) look out for words "initial" "start"

$t = 0$

$M = 30 \times 2^{-0.001(0)}$

$M = 30 \times 2^{0}$

$M = 30 \times 1$

$M = 30$

So 30grams

(ii) Find t, $M = 10$

$M = 30 \times 2^{-0.001t}$

$10 = 30 \times 2^{-0.001t}$

$\frac{10}{30} = \frac{30 \times 2^{-0.001t}}{30}$

$\frac{1}{3} = 2^{-0.001t}$

$\log_2\left(\frac{1}{3}\right) = -0.001t$

$-1.584962501 = -0.001t$

$1584.96 = t$

1584.96 years.

1585 years.

(iii) find t, $M = 0.01(30)$

$M = 0.3$

$M = 30 \times 2^{-0.001t}$

$0.3 = 30 \times 2^{-0.001t}$

$\frac{0.3}{30} = \frac{30 \times 2^{-0.001t}}{30}$

$0.01 = 2^{-0.001t}$

$\log_2 0.01 = -0.001t$

$-6.64385619 = -0.001t$

$6643.856 = t$

6643.86 yrs

6644 yrs

By choosing a suitable base, solve the following equation for x.

$$\log_5 x - 1 = 6 \log_x 5$$

$\log_x 5 = \frac{\log_5 5}{\log_5 x} = \frac{1}{\log_5 x}$

$\log_5 x - 1 = 6\left(\frac{1}{\log_5 x}\right)$

Let $y = \log_5 x$

$y - 1 = 6\left(\frac{1}{y}\right)$

$y - 1 = \frac{6}{y}$

$y(y) - y(1) = y\left(\frac{6}{y}\right)$

$y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y+2)(y-3) = 0$

$y = -2 \qquad y = 3$

$\log_5 x = -2 \qquad \log_5 x = 3$

$5^{-2} = x \qquad 5^3 = x$

$\frac{1}{5^2} = x \qquad 125 = x$

$\frac{1}{25} = x$

⇒ By letting $2^x = y$, solve: $2^{x+1} + 2(2^{-x}) - 5 = 0$

$$(2^x)(2^1) + 2\left(\frac{1}{2^x}\right) - 5 = 0$$

$$(y)(2) + 2\left(\frac{1}{y}\right) - 5 = 0$$

$$2y + \frac{2}{y} - 5 = 0$$

$$(y)(2y) + (\cancel{y})\left(\frac{2}{\cancel{y}}\right) - (y)(5) = (y)(0)$$

$$2y^2 + 2 - 5y = 0$$

$$2y^2 - 5y + 2 = 0$$

$2 \times 2 = 4$ $\quad -4, -1$

$$2y^2 - 4y - y + 2 = 0$$

$$2y(y - 2) - 1(y - 2) = 0$$

$$(2y - 1)(y - 2) = 0$$

$2y = 1$ or $y = 2$

$y = \frac{1}{2}$ $\qquad$ $2^x = 2$

$2^x = \frac{1}{2}$ $\qquad$ $\log_2 2 = x$

$\log_2\left(\frac{1}{2}\right) = x$ $\qquad$ $\underline{\underline{1 = x}}$

$\underline{\underline{-1 = x}}$