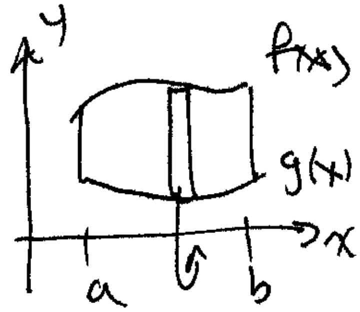
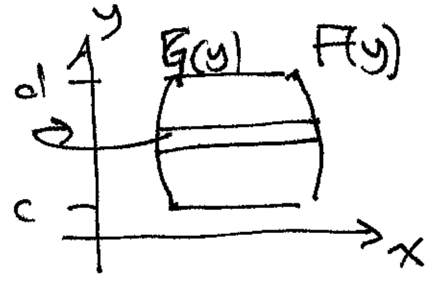
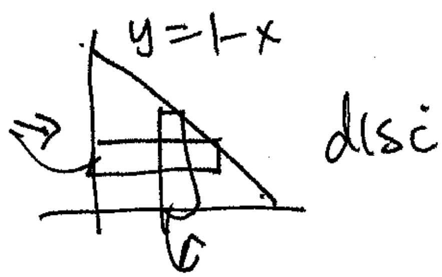
Volumes of Revolution – Disc Method

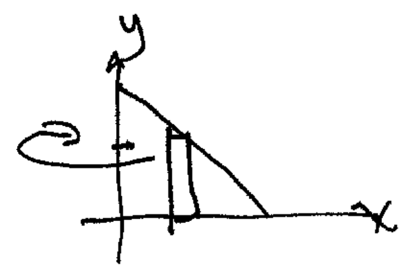




$V = \pi \int_a^b f^2(x) - g^2(x)\, dx$ $\qquad$ $V = \pi \int_c^d F^2(y) - G^2(y)\, dy$

Now we use another method to find volumes of revolution. Consider $f(x) = 1 - x$ on $[0,1]$





$$V = \pi \int_0^1 (1-y)^2\, dy$$

$$= \pi \left. y - y^2 + \frac{y^3}{3} \right|_0^1$$

$$= \pi \left(1 - 1 + \frac{1}{3}\right) = \pi/3$$

Here we have a hollow cylinder

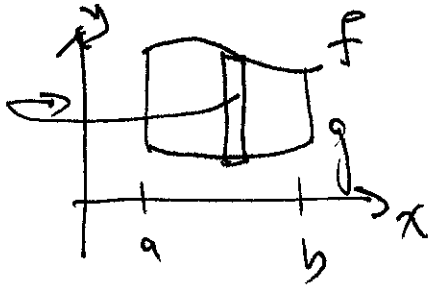
so $dV = 2\pi r h\, dx$

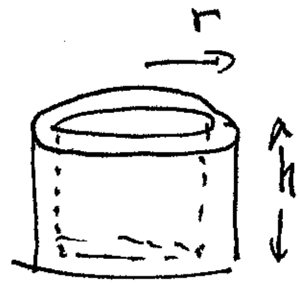
$= 2\pi x(1-x)\, dx$

$$V = 2\pi \int_0^1 (x - x^2)\, dx = 2\pi \left. \left(\frac{x^2}{2} - \frac{x^3}{3}\right) \right|_0^1$$

$$= \pi/3$$

In general

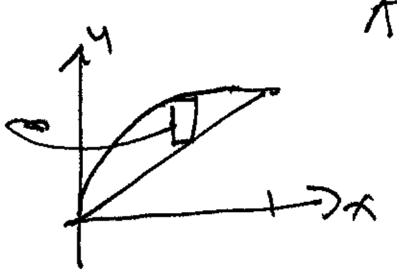




$$V = 2\pi \int_a^b x\,(f(x) - g(x))\,dx$$

This is known as the "shell" method

Ex Find the volume of the solid when the region between $y = \sqrt{x}$ & $y = x$ is revolved around the y axis



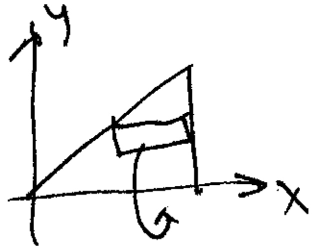
Again the intersection pts are $x = 0, 1$ so

$$V = 2\pi \int_0^1 x(\underbrace{\sqrt{x}}_{\text{top curve}} - \underbrace{x}_{\text{bottom curve}})\,dx = 2\pi \int_0^1 (x^{3/2} - x^2)\,dx$$

$$= 2\pi\left(\frac{2x^{5/2}}{5} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = 2\pi\left(\frac{6-5}{15}\right) = \frac{2\pi}{15}$$ we saw this earlier

We can also revolve about the y axis using shells. Ex the radius now is in the y direction



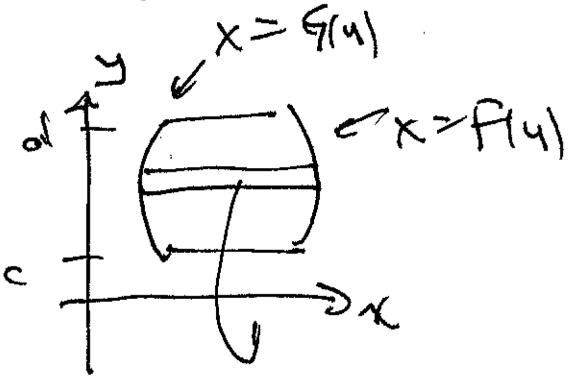
$$V = 2\pi \int_0^1 y\,(1-y)\,dy$$

x=1 right curve

x=y left curve

$$= 2\pi \int_0^1 (y-y^2)\,dy = 2\pi\left(\frac{y^2}{2}-\frac{y^3}{3}\Big|_0^1\right) = \frac{2\pi\cdot 1}{6} = \pi/3$$

In general

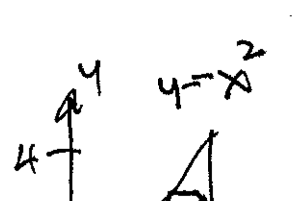


$$V = 2\pi \int_c^d y\,(F(y) - G(y))\,dy$$

right curve

~~right~~ left curve

Ex Revolve the region $y=x^2$ & $y=0$, $x=2$ about x axis using shells

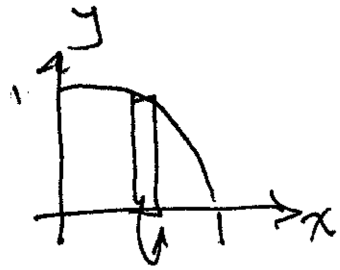


$$V = 2\pi \int_0^4 y\,(2-\sqrt{y})\,dy = 2\pi\int_0^4 (2y - y^{3/2})\,dy$$

$$= 2\pi\left(y^2 - \frac{2y^{5/2}}{5}\Big|_0^4\right) = 2\pi\left(16 - \frac{2\cdot 32}{5}\right)$$

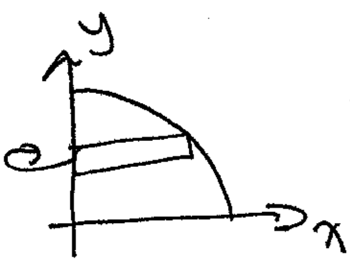
$$= 2\pi\cdot\frac{16}{5} = 32\pi/5$$

Ex Revolve the region bound by $y=\sqrt{1-x^2}$ about the x & y axis using disc's & shells



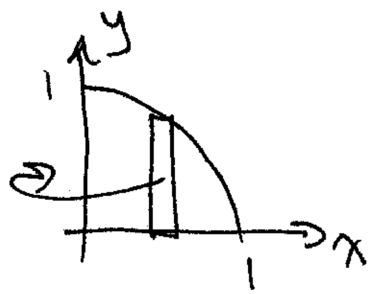
$$V=\pi\int_0^1\left(\sqrt{1-x^2}\right)^2dx - \pi\int_0^1(1-x^2)\,dx$$

$$=\pi\left(x-\frac{x^3}{3}\Big|_0^1\right) = \frac{2\pi}{3}$$



$$V=\pi\int_0^1\left(\sqrt{1-y^2}\right)^2dy = \pi\int_0^1(1-y^2)\,dy$$

$$=\pi\; y-\frac{y^3}{3}\Big|_0^1 = \frac{2\pi}{3}$$



$$V=2\pi\int_0^1 x\sqrt{1-x^2}\,dx$$

$u=1-x^2$

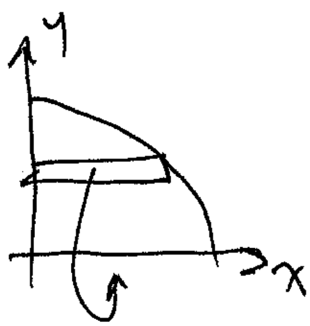
$du=-2x\,dx$

$x=0 \quad u=1$

$x=1 \quad u=0$

$$=\pi\int_1^0 -u^{1/2}\,du$$

$$=\pi\int_0^1 u^{1/2}\,du = \pi\frac{2u^{3/2}}{3}\Big|_0^1 = \frac{2\pi}{3}$$



$$V=2\pi\int_0^1 y\sqrt{1-y^2}\,dy$$ same integral except $x \leftrightarrow y$

$$=2\pi/3$$