Hierarchical Assembly of Leader-Asymmetric, Single-Leader Networks

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Abstract— **The connection between leader-asymmetry and controllability in controlled agreement networks provides a topological, necessary condition for controllability. In this paper we investigate how to produce hierarchical networks that, at each level in the hierarchy, exhibit the leader-asymmetry properties. Graph grammars are moreover provided for assembling the leader-asymmetric networks of any size.**

I. Introduction

In this paper, we investigate how to construct network topologies in a hierarchical manner in such a way that they are amenable to external control. In particular, the networks will be comprised of a collection of nodes whose cohesion is ensured through agreement-based interaction rules. These networks can moreover be controlled by injecting control signals at particular input-nodes (so-called leader-nodes) in the networks. The control of such multi agent systems has received considerable attention during the last decade and several results have been presented regarding the analysis of the underlying structure and characteristics of these distributed coordination systems, e.g. [1],[2],[3],[4].

One key question when trying to design controllers for such networks is whether or not they are even controllable in the first place. Controllability issues in these types of networked systems was first discussed in [5], where conditions for controllability were given in terms of the eigenvectors of the graph Laplacian. Later, a more topological exploration of the controllability properties in such leader-follower networks was given in [6], presenting a sufficient condition for a network to be uncontrollable in the case of single leader case using graph symmetries. These concepts were extended in [7] and through the use of equitable partitions. In [8], a graph theoretic discussion of the controllable subspace of an uncontrollable network was given. Some other results related to controllability of leader-follower systems were presented in [9].

A key concept when studying the controllability of networked systems that has emerged is the notion of an external, equitable partition. Such a partition groups together nodes into cells, and members of the same cell have been shown to converge asymptotically to the same subspace. As such, a necessary condition for a network to be completely controllable is that no such cells exist that share more than a single node. We will refer to such networks as *leader-asymmetric* and this paper addresses the construction of leader-asymmetric networks through the inter-connections of multiple sub-networks that are themselves leader-asymmetric. This process of inter-connecting networks leads, in turn, to a hierarchical structure and the relationship between the leader-asymmetry properties at each stage of the hierarchy and the overall network is presented.

Moreover, these results are then used to provide graph grammar rules for the self-assembly of individual nodes into a leader-asymmetric network of any size with a single leader. Also, the maximum distance of any follower node from the leader in these resulting networks is also given. The reason for defining graph grammars for this assembly task is that they have been used to model self-assembly processes involving a large number of mobile agents in a natural and direct manner, as shown in [10],[11].

This paper has two major parts. One is related to the hierarchical construction of leader-asymmetric, single-leader networks and the other is related to the graph grammars for the self assembly of such networks. Our presentation starts with a system description in Section III. In Section IV, we review some results regarding the controllability and leader-asymmetry of single leader networks from graph theoretic point of view. In Section V, we present results related to the leader-asymmetry of interconnected and hierarchical networks. Section VI reviews the basics of graph grammar constructions, and Section VII presents the rule sets for self assembly of leader-asymmetric, single-leader systems.

II. System Description

In this section, we show how can we construct a large single-leader network, by connecting together smaller single-leader networks in a hierarchical way. The main idea is to grow the network by connecting together individual networks at different stages, where, smaller single leader networks constitute the first stage of this process. At the next stage, the leaders of these smaller subnetworks are connected together with an external node called the super-leader. This super-leader serves as the external input to our system. Throughout this paper, by a graph, we mean, an undirected graph with no loops and multiple edges between the vertices.

Consider n identical leader-follower networks, i.e. networks where the control input is injected at the leader-node. Each of these networks $\mathcal{G}^{(i)}$, where $i \in \{1,2,\ldots,n\}$, has a single leader and m followers. The hierarchical construction is now to connect together the leaders of all $\mathcal{G}^{(i)}$ via another leader-follower network $\mathcal{G}^{(l)}$, where the leaders take on follower roles, and a new node x_{l_s} takes on the leader role. x_{l_s} is a super leader and it is also a leader node of $\mathcal{G}^{(l)}$.

Now, the overall network $\mathcal{G}$ that is obtained by connecting $\mathcal{G}^{(1)}, \mathcal{G}^{(2)}, \ldots, \mathcal{G}^{(n)}$ together via their leaders to an external

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node x_{l_s}, is also a leader-follower network with a single leader x_{l_s} and all other nodes being followers. This construction is illustrated in Fig. 1

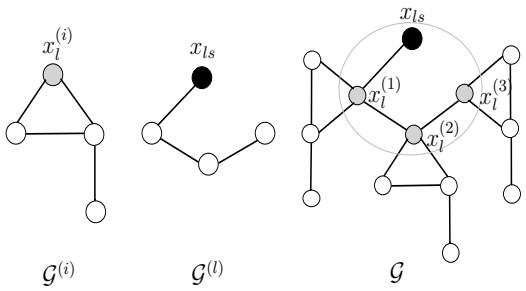

Fig. 1. $\mathscr{G}^{(i)}$ is a leader-follower network with a single leader $x_l^{(i)}$ and $m=3$ followers. $n=3$ such networks are connected together via their leaders through another leader-follower network $\mathscr{G}^{(l)}$ where an external node x_{ls} takes on a leader role. The resulting inter-connected network $\mathscr{G}$ is also a leader-follower network with x_{ls} as a leader and all other nodes being followers.

What we would like to understand is how certain key properties associated with the controllability of the individual networks are inherited by the new inter-connected network $\mathscr{G}$. This is the topic of the next sections that start with a discussion of what the key topological properties are.

III. Equitable Partitions and Leader-Asymmetric, Single Leader Networks.

In this section we will review the scope of equitable partitions in examining the controllability of single leader networks and state some results from [7],[8] and [9]. These results will connect the study of leader-asymmetry with that of controllability (or at least with uncontrollability), which is a connection that will be pursued throughout the remainder of this paper.

Definition 3.1: (External Equitable Partition): A partition π of nodes $\mathscr{X}$ of a graph $\mathscr{G}$, with cells $C_1, C_2, \cdots, C_r$ is said to be an *external equitable partition* if each node in C_i has the same number of neighbors in C_j, for all $i, j \in \{1,2,\cdots,r\}, i \neq j$, with $r = \mid \pi \mid$, which denotes the cardinality of partition.

Definition 3.2: (Non-Trivial External, Equitable Partition): An external equitable partition in which at least one cell has more than one nodes is a non-trivial external, equitable partition.

Definition 3.3: (Leader-Invariant External, Equitable Partition (LEP)): The LEP is a partition $\pi_M = \pi_F \bigcup \pi_L$, where $\pi_F = \{C_1^M, C_2^M, \cdots, C_s^M\}$ is the external equitable partition of the follower nodes such that the cardinality of π_F is minimal (i.e. has the fewest cells), and the leader L belongs to the singleton cell $C_{s+1}^M = \{L\}$ of the partition $\pi_L = \{C_{s+1}^M\}$.

Fact 1: Every leader follower network has a unique LEP [9].

Definition 3.4: (Leader-Asymmetric Single-Leader Network): A leader-follower network is said to be a leader-asymmetric if its LEP is trivial i.e. every cell in its LEP is a singleton cell.

Fact 2: A single leader network executing the controlled agreement dynamics is completely controllable only if it is leader-asymmetric [9]. (see Fig. 2)

In light of the second fact, we will shift our focus from controllability to the leader-asymmetry throughout the remainder of this paper. The reason for this shift is that the leader-asymmetry is a purely topological condition, while controllability is not. It should be noted that it is just a necessary condition for controllability and no topological necessary and sufficient condition has, as of yet, been found for the single-leader, controlled agreement dynamics.

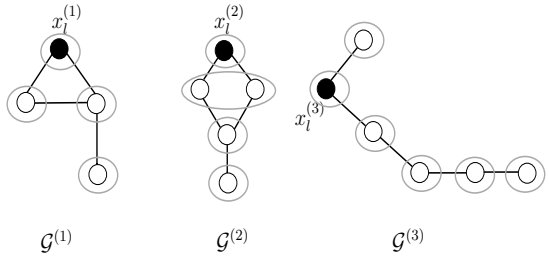

Fig. 2. $\mathscr{G}^{(1)}$ has a trivial LEP, so it is leader-asymmetric, single leader network. It is also completely controllable with $x_l^{(1)}$ as its leader. $\mathscr{G}^{(2)}$ is not leader-asymmetric as it has a non trivial LEP. $\mathscr{G}^{(3)}$ has a trivial LEP, so it is leader-asymmetric but it is not completely controllable with $x_l^{(3)}$ as its leader.

IV. Hierarchical Leader-Asymmetric, Single Leader Networks.

Definition 4.1: (Connection Network and Interconnected Network) Consider n leader-follower networks $\mathscr{G}^{(1)}, \mathscr{G}^{(2)}, ..., \mathscr{G}^{(n)}$, each with a single leader $x_l^{(1)}, x_l^{(2)}, ..., x_l^{(n)}$ respectively[1], then $\mathscr{G}$ is a network obtained by connecting $\mathscr{G}^{(1)}, \mathscr{G}^{(2)}, ..., \mathscr{G}^{(n)}$ together via their leaders only to an external node called super leader x_{l_s}, through a *connection network* $\mathscr{G}^{(l)}$, as discussed in the Section II. The resulting $\mathscr{G}$ is said to be an *interconnected network*, which is also a leader-follower network with a single leader x_{l_s} and all other nodes being followers.

One question one might ask is whether or not this type of construction preserves certain desirable properties. In this paper, we will focus on the issue of leader-asymmetry as defined in the Definition 3.4. However, we start with the question of controllability and see that this property is in fact, not preserved when controllable networks are interconnected.

Lemma 4.1: Let $\mathscr{G}$ be an interconnected single-leader network as per Definition 4.1. If the individual networks $\mathscr{G}^{(1)}, \mathscr{G}^{(2)}, ..., \mathscr{G}^{(n)}$ are completely controllable with respect to their respective leaders, then complete controllability of the connection network $\mathscr{G}^{(l)}$ is neither a necessary nor a sufficient condition for the complete controllability of $\mathscr{G}$.

Proof: (Counter example)- Let $\mathscr{G}^{(1)}$ and $\mathscr{G}^{(2)}$ be two completely controllable single-leader networks with $x_l^{(1)}$ and $x_l^{(2)}$ as leaders respectively, as shown in the Fig. 3. *Interconnected network* $\mathscr{G}$ is obtained by connecting $\mathscr{G}^{(1)}$ and $\mathscr{G}^{(2)}$ through a completely controllable *connection network* $\mathscr{G}^{(l)}$.

[1]Note that we will, throughout this paper, use $x^{(i)}$ to denote the state associated with node i, but we will also use it as shorthand to denote the node itself, whenever this is clear from the context.

The resulting *interconnected network* $\mathscr{G}$ is not completely controllable with x_{l_s}, as its *LEP* is non-trivial. (see Fig. 3.)

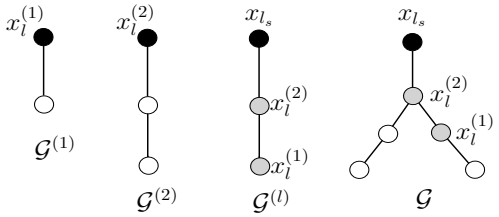

Fig. 3. Complete controllability of *connection network* $\mathscr{G}^{(l)}$ is not a sufficient condition for controllability of *interconnected network* $\mathscr{G}$.

Now, let us connect the leaders of same $\mathscr{G}^{(1)}$ and $\mathscr{G}^{(2)}$ through another connection network $\mathscr{G}^{(l)}$ that is symmetric with respect to x_{l_s}, and hence uncontrollable. The resulting $\mathscr{G}$ is completely controllable with respect to x_{l_s} even though $\mathscr{G}^{(l)}$ is uncontrollable. (see Fig. 4.) ■

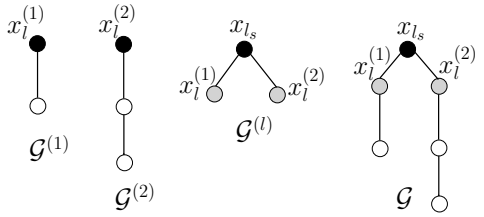

Fig. 4. Complete controllability of *connection network* $\mathscr{G}^{(l)}$ is not a necessary condition for controllability of *interconnected network* $\mathscr{G}$.

This shows that the complete controllability of the individual networks does not ensure the complete controllability of the interconnected network.

Lemma 4.2: Let $\mathscr{G}^{(1)}$ be a path network with one of the end nodes as a leader $x_l^{(1)}$ and $\mathscr{G}^{(2)}$ be any leader-asymmetric single leader network with a leader $x_l^{(2)}$. Let $\mathscr{G}$ be a network obtained by connecting the second end node of $\mathscr{G}^{(1)}$ with $x_l^{(2)}$, then $\mathscr{G}$ is also a leader-asymmetric, single leader network with $x_l^{(1)}$ as a leader.

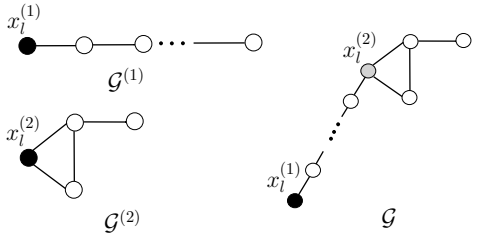

Fig. 5. Path network connected with a leader-asymmetric, single leader network gives leader-asymmetric $\mathscr{G}$.

Proof: A path network with a terminal node as the only leader, is completely controllable and hence, in $\pi_{\mathscr{G}}$, cells containing the nodes of $\mathscr{G}^{(1)}$ will be singletons. Since $x_l^{(2)}$ is the only node of $\mathscr{G}^{(2)}$, connected with any node of $\mathscr{G}^{(1)}$, so a cell containing $x_l^{(2)}$ is also singleton. Since $\mathscr{G}^{(2)}$ is itself leader asymmetric and $x_l^{(2)}$ is in a singleton cell in $\pi_{\mathscr{G}}$, so all other nodes of $\mathscr{G}^{(2)}$ will also be in singleton cells. Thus, giving a trivial $\pi_{\mathscr{G}}$. Hence $\mathscr{G}$ is leader-asymmetric. (see Fig. 5.) ■

Lemma 4.3: Let $\mathscr{G}$ be an interconnected single leader network as per Definition 4.1, with all subnetworks $\mathscr{G}^{(i)}$ being leader-asymmetric, then, $\mathscr{G}$ is also leader-asymmetric iff cells containing $x_l^{(i)}$ in $\mathscr{G}$ are singletons.

Proof:($\Rightarrow$) If $\mathscr{G}$ is leader-asymmetric, then it has a trivial LEP $\pi_{\mathscr{G}}$ by definition with all cells being singletons.

($\Leftarrow$) Let us assume for the sake of contradiction that in $\pi_{\mathscr{G}}$, cells containing $x_l^{(i)}$'s are singletons but $\mathscr{G}$ is not leader-asymmetric. Then, there must be a cell C^* in $\pi_{\mathscr{G}}$ containing more than one nodes that either belong to (a) same subnetwork $\mathscr{G}^{(i)}$ or (b) different subnetworks. (a) is not possible as each subnetwork is itself leader asymmetric and each $x_l^{(i)}$ is in a singleton cell in $\pi_{\mathscr{G}}$. For (b), since followers of one subnetwork $\mathscr{G}^{(i)}$ are not connected to the leader of another subnetwork, so nodes in a cell C^* containing followers of different subnetworks, can never have same node to cell degree with other cells, as required by LEP construction. So, (b) is also not possible. ■

Theorem 4.4: If $\mathscr{G}$ is an interconnected network as per Definition 4.1 and $\mathscr{G}^{(1)},\mathscr{G}^{(2)},...,\mathscr{G}^{(n)}$ are identical, leader-asymmetric, single leader networks, then leader-asymmetry of the connection network $\mathscr{G}^{(l)}$ with x_{ls} as a leader, is a sufficient condition for the interconnected network $\mathscr{G}$ to be leader-asymmetric with the same leader x_{ls}.

Proof: Let $\mathscr{X}$ be a set, containing the leader nodes of $\mathscr{G}^{(i)}$ i.e. $\mathscr{X} = \{x_l^{(1)}, x_l^{(2)}, \cdots, x_l^{(n)}\}$. Also, let $\mathscr{X}_{dir} = \{x_l^{(i)} \in \mathscr{X} : \ x_l^{(i)}$ is directly connected to $x_{l_s}\}$. Similarly, $\mathscr{X}_{not\ dir} = \{x_l^{(i)} \in \mathscr{X} : \ x_l^{(i)}$ is not directly connected to $x_{l_s}\}$. Note that, $\mathscr{X} = \mathscr{X}_{dir} \cup \mathscr{X}_{not\ dir}$

For proving the above theorem, we will prove the following claims first.

Claim 1: In $\pi_{\mathscr{G}}$, every $x_d \in \mathscr{X}_{dir}$ is in a singleton cell.

Proof: Let C_{l_s} be a cell in $\pi_{\mathscr{G}}$, containing the super leader x_{l_s}. Now assume for the sake of contradiction that there exists $x_d \in \mathscr{X}_{dir}$ not contained in a singleton cell C_d. Then, this C_d can only contain another $x_{d'} \in \mathscr{X}_{dir}$ as they are the only nodes in $\pi_{\mathscr{G}}$ directly connected to x_{l_s} (and C_{l_s}). This requires $\mathscr{G}$ to be symmetric about x_{l_s}. Since all $\mathscr{G}^{(i)}$'s are identical and leader-asymmetric, so $\mathscr{G}$ can be symmetric about x_{l_s} iff the connection network $\mathscr{G}^{(l)}$ is symmetric about x_{l_s}. But $\mathscr{G}^{(l)}$ is leader-asymmetric by construction and so, not symmetric about x_{l_s}. Thus, $\mathscr{G}$ is also not symmetric about x_{l_s} and hence, our assumption is not true, thus, proving the claim.

Claim 2: In $\pi_{\mathscr{G}}$, every $x_{nd} \in \mathscr{X}_{not\ dir}$ is also in a singleton cell.

Proof: Let us assume for the sake of contradiction that there exist $x_{nd} \in \mathscr{X}_{not\ dir}$ that is not in a singleton cell C_{nd} in $\pi_{\mathscr{G}}$. Also, it is directly connected to some $x_d \in \mathscr{X}_{dir}$. Then there is one of the following possibilities that (a) C_{nd} also has some $x_{d'} \in \mathscr{X}_{dir}$. (b) C_{nd} also has a follower node of subnetwork whose leader is x_d or (c) C_{nd} also has some other $x_{nd} \in \mathscr{X}_{not\ dir}$. (a) is not possible by claim 1. (b) is

not possible as all subnetworks $\mathscr{G}^{(i)}$ are identical and also leader-asymmetric, so there will always be a follower node in the subnetwork of leader x_d that can never be contained in a valid cell in $\pi_{\mathscr{G}}$.

For (c), let us assume that C_{nd} also contain $x_{nd'} \in \mathscr{X}_{not\ dir}$ along with x_{nd}. Since, x_{nd} is directly connected to x_d that is in a singleton cell C_d, this requires $x_{nd'}$ to be directly connected to x_d also. Now, to maintain the same node to cell degree condition for a valid $\pi_{\mathscr{G}}$, the leader in $\mathscr{X}_{not\ dir}$ that is directly connected to x_{nd} will be contained in a cell along with some other leader that is directly connected to $x_{nd'}$. This will continue until we get a cell containing $x_{nd_\alpha} \in \mathscr{X}_{not\ dir}$ and $x_{nd_\beta} \in \mathscr{X}_{not\ dir}$. Now there must be another leader in $\mathscr{X}_{not\ dir}$ that should be directly connected to either one of x_{nd_α} or x_{nd_β} but not both, as otherwise the connection network $\mathscr{G}^{(l)}$ cannot be leader-asymmetric. Let that remaining leader be $x_{nd_{last}}$ and without loss of generality, it is connected to x_{nd_α}. Then for a valid $\pi_{\mathscr{G}}$, the cell $C_{nd_{last}}$ containing $x_{nd_{last}}$ must also contain a follower node of subnetwork whose leader is x_{nd_β} and that follower node should be directly connected to x_{nd_β}. Since all subnetworks are identical and leader-asymmetric with the the same number of follower nodes, so a follower node in the subnetwork of $x_{nd_{last}}$ will always be left that can never be contained in a valid cell in $\pi_{\mathscr{G}}$. So our assumption is not valid and the claim is true.

From claims 1 and 2, we get that in LEP of $\mathscr{G}$, all leaders $x_l^{(i)}$ of subnetworks $\mathscr{G}^{(i)}$ will be contained in singleton cells. Thus, from Lemma 4.3, we get that $\mathscr{G}$ is leader-asymmetric. ■

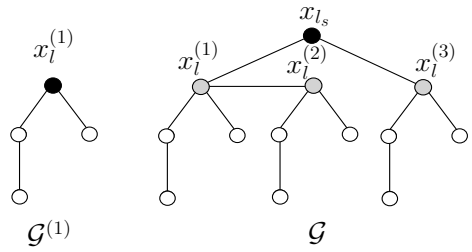

Fig. 6. An example illustrating Theorem 4.4.

Theorem 4.5: Let $\mathscr{G}^{(1)},\mathscr{G}^{(2)},...,\mathscr{G}^{(n)}$ be identical and leader-asymmetric single leader networks with leaders $x_l^{(1)},x_l^{(2)},...,x_l^{(n)}$ respectively. Let $\mathscr{G}$ be a network obtained by interconnecting the leaders $x_l^{(i)}$ through a network $\mathscr{G}^{(l)}$. If $\mathscr{G}^{(l)}$ is leader-asymmetric with one of $x_l^{(i)}$ say $x_l^{(*)}$, then $\mathscr{G}$ is also leader-asymmetric with a single leader $x_l^{(*)}$.

Proof: Let $\mathscr{X} = \{x_l^{(1)},x_l^{(2)},\cdots,x_l^{(n)}\}$. Also, assume that there exists $x_l^{(*)} \in \mathscr{X}$ such that $\mathscr{G}^{(l)}$ is leader-asymmetric with $x_l^{(*)}$. Also $x_l^{(*)}$ is a leader of a subnetwork $\mathscr{G}^{(*)}$.

Let us do the partition of $\mathscr{X}$ as $\mathscr{X} = \mathscr{X}_{dir} \bigcup \mathscr{X}_{not\ dir} \bigcup \{x_l^{(*)}\}$, where, $\mathscr{X}_{dir} = \{x_l^{(i)} \in \mathscr{X} : x_l^{(i)}$ is directly connected to $x_l^{(*)}\}$. Also, $\mathscr{X}_{not\ dir} = \{x_l^{(i)} \in \mathscr{X} : x_l^{(i)}$ is not directly connected to $x_l^{(*)}\}$.

Claim: In $\pi_{\mathscr{G}}$, the follower nodes of subnetwork $\mathscr{G}^{(*)}$ will be contained in singleton cells.

Proof: Let $\mathscr{V}$ be a set of follower nodes of $\mathscr{G}^{(*)}$ and $\mathscr{V} = \mathscr{V}_d \bigcup \mathscr{V}_{nd}$, where $\mathscr{V}_d \subseteq \mathscr{V}$ is a subset containing those followers of $\mathscr{G}^{(*)}$ that are directly connected to $x_l^{(*)}$ and $\mathscr{V}_{nd} \subseteq \mathscr{V}$ is a subset containing the followers of $\mathscr{G}^{(*)}$ not directly connected to $x_l^{(*)}$.

Firstly, we show that, in $\pi_{\mathscr{G}}$, any $v_d \in \mathscr{V}_d$ will be in a singleton cell. For the sake of contradiction, let us assume that there exists $v_{d_1} \in \mathscr{V}_d$ such that cell $C_{v_{d_1}}$ containing v_{d_1} is not singleton. Then $C_{v_{d_1}}$ will also contain one of the following along with v_{d_1}, (a) some $v_{nd} \in \mathscr{V}_{nd}$, (b) some other $v_d \in \mathscr{V}_d$, (c) some $x_{nd} \in \mathscr{X}_{not\ dir}$, (d) some follower node of a subnetwork $\mathscr{G}^{(i)}$ where $\mathscr{G}^{(i)} \neq \mathscr{G}^{(*)}$. or (e) some $x_d \in \mathscr{X}_{dir}$. Out of these (a),(c) and (d) are not possible as none of the nodes in these options is directly connected to $x_l^{(*)}$ while $v_{d_1} \in C_{v_{d_1}}$ is directly connected to $x_l^{(*)}$. Now, since $\mathscr{G}^{(*)}$ is leader-asymmetric, so it is not symmetric about $x_l^{(*)}$, hence (b) is also not possible. For (e), assume that $C_{v_{d_1}}$ contains v_{d_1} and some $x_{d_1} \in \mathscr{X}_{dir}$, where x_{d_1} is a leader of a subnetwork $\mathscr{G}^{(d_1)}$. Then $v_{nd_1} \in \mathscr{V}_{nd}$, where v_{nd_1} is directly connected to v_{d_1}, and a follower node of subnetwork $\mathscr{G}^{(d_1)}$ say $v_{f_1}^{(d_1)}$ directly connected to x_{d_1}, must also be contained in the same cell due to the construction rules of $\pi_{\mathscr{G}}$. Similarly, in the next step, $v_{nd_2} \in \mathscr{V}_{nd}$ where v_{nd_2} is directly connected to v_{nd_1}, and a follower node of $\mathscr{G}^{(d_1)}$ directly connected to $v_{f_1}^{(d_1)}$, must also be in a same cell. This will continue and since all subnetworks $\mathscr{G}^{(i)}$ are identical, so we will always be left with a follower node in $\mathscr{G}^{(d_1)}$ that cannot be contained in a valid cell in $\pi_{\mathscr{G}}$. So,(e) is also not possible, and every $v_d \in \mathscr{V}_d$ will be in a singleton cell in $\pi_{\mathscr{G}}$.

Note that followers of one subnetwork $\mathscr{G}^{(i)}$ are not connected with the leaders or followers of another subnetwork. Also, in $\mathscr{G}^{(*)}$, every $v_d \in \mathscr{V}_d$ will be in a singleton cell in $\pi_{\mathscr{G}}$. Since $\mathscr{G}^{(*)}$ is also leader-asymmetric, so these facts will directly imply that every $v_{nd} \in \mathscr{V}_{nd}$ will also be in a singleton cell. This proves our claim.

If we remove the follower nodes of $\mathscr{G}^{(*)}$ from $\mathscr{G}$, we get $\tilde{\mathscr{G}}$, where $\tilde{\mathscr{G}}$ exactly satisfies the conditions in Theorem 4.4 with $x_l^{(*)} = x_{l_s}$, thus, $\pi_{\tilde{\mathscr{G}}}$ is trivial, with all nodes being in singleton cells. Now adding the follower nodes of $\mathscr{G}^{(*)}$ to $\tilde{\mathscr{G}}$ will give us $\mathscr{G}$. By the above claim, we know that follower nodes of $\mathscr{G}^{(*)}$ can never be a cause of non trivial $\pi_{\mathscr{G}}$ if $\mathscr{G}^{(*)}$ is identical to the other subnetworks $\mathscr{G}^{(i)}$. Combining these facts, we conclude that $\pi_{\mathscr{G}}$ is also trivial and hence, $\mathscr{G}$ is leader-asymmetric with $x_l^{(*)}$ as a leader. ■

V. Graph Grammar Preliminaries.

In this section, we will review the basics of graph grammars approach to model the task of assembling large number of self controlled parts into a prescribed formation. We refer the readers to [10] and [11] for more details about this topic.

Definition 5.1: (Rule): A rule is a pair of graphs $r = (G_a, G_b)$ that changes the edge set $E(G_a)$ of G_a to $E(G_b)$ to give G_b while keeping the vertex set constant, i.e. $V(G_a) = V(G_b)$. The *size* of r is $V(G_a) = V(G_b)$.

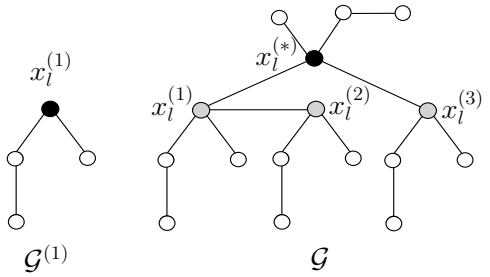

Fig. 7. An example illustrating Theorem 4.5.

Definition 5.2: (Rule Set or Grammar): A rule set (or grammar) Φ is a set of rules that defines a concurrent algorithm for a group of individual nodes to follow.

Definition 5.3: (System): A system is a pair (G_0, Φ) where G_0 is an initial graph of the system and Φ is a set of rules applied on G_0.

Definition 5.4: (Trajectory): A trajectory of a system (G_0, Φ) is a (finite or infinite) sequence

$$G_0 \xrightarrow{(r_1,h_1)} G_1 \xrightarrow{(r_2,h_2)} G_2 \xrightarrow{(r_3,h_3)} \cdots$$

If the sequence is finite, then there exists a terminal graph where no rule in Φ is applicable. We denote a trajectory of a system by τ and the set of all such trajectories by $\mathscr{T}(G_0, \Phi)$. Also, we use the notation τ_j to denote the j^{th} graph in the trajectory τ_j.

VI. Graph Grammars for Producing Leader-Asymmetric Single Networks

In this section we will show how graph grammars can be used to produce leader-asymmetric, single leader networks of any size in a decentralized way. These simple rules can be used to produce subnetworks of any size and then using the previous results, we can construct bigger networks out of them that are also leader-asymmetric with a single leader. So we can state our goal as, *Construct a rule set Φ for a system $(\mathscr{G}_o, \Phi)$ with $\mathscr{G}_o$ as a set of isolated nodes, such that trajectory of a system, $\tau \in \mathscr{T}(\mathscr{G}_o, \Phi)$, is a finite sequence with a terminal graph as a set of leader-asymmetric, single leader networks with p nodes.* We call the resulting leader-asymmetric networks with p nodes and a single leader as a *crystal*. We will also provide *maximum leader to node distance, d* in that crystal, resulting from $(\mathscr{G}_o, \Phi)$. We also use the notation $|\mathscr{G}|$ to denote the cardinality of the vertex set of the graph $\mathscr{G}$.

A. Rules for Crystals of Size $p = 2^n, n \geq 1$

Consider the following rule set Φ_A

$$\Phi_A = \begin{cases} (r_0) & a \quad a \rightharpoonup \ell_1 \text{—} c \\ (r_1) & \ell_i \quad \ell_i \rightharpoonup \ell_{i+1} \text{—} c \quad 1 \leq i \leq n-1 \end{cases}$$

Claim: Φ_A gives leader-asymmetric, single leader crystals of size $p = 2^n$.

Proof: Let τ^A be a trajectory obtained by Φ_A and τ_j is the j^{th} graph in this trajectory. Then τ_0 is a path graph with a single node c and a single leader ℓ_1. τ_1 is obtained by connecting two τ_0 via their leaders and making one of them as a new leader ℓ_2. By Theorem 4.5, τ_1 is leader-asymmetric with leader ℓ_2 as τ_0 is leader-asymmetric. Also $|\tau_j| = 2\,|\tau_{j-1}|$. This continues until we get a terminal graph with τ_{n-1} which is infact leader-asymmetric with a single leader ℓ_n and $|\tau_{n-1}| = p = 2^n$. ∎

Here maximum leader to node distance, $d = n$.

B. Rules for Crystals of Size $p = k(2)^n, k \geq 3, n \geq 0$

Consider the following rule set Φ_B

$$\Phi_B = \begin{cases} (r_0) & a \quad a \rightharpoonup \ell_1 \text{—} b_1 \\ (r_1) & b_i \quad a \rightharpoonup c \text{—} b_{i+1} \quad 1 \leq i \leq (k-3) \\ (r_2) & b_{k-2} \quad a \rightharpoonup c \text{—} c \\ (r_3) & \ell_j \quad \ell_j \rightharpoonup \ell_{j+1} \text{—} c \quad 1 \leq j \leq n \end{cases}$$

Claim: Φ_B gives leader-asymmetric crystals of size $p = k(2)^n$ with single leader.

Proof: Proof is exactly like the proof of Φ_A with the only addition that in the initial steps, the first three rules r_0, r_1, r_2 are creating a path graph τ_{k-2} with $|\tau_{k-2}| = k$ with a single leader ℓ_1. ∎

Maximum leader to node distance, *d = (n+k)-1* in this case.

C. Rules for Crystals of Size $p = k(2)^n + 1, k \geq 3, n \geq 0$

Consider the following rule set Φ_C

$$\Phi_C = \begin{cases} (r_0) & a \quad a \rightharpoonup \varepsilon_1 \text{—} b_1 \\ (r_1) & b_i \quad a \rightharpoonup c \text{—} b_{i+1} \quad 1 \leq i \leq (k-3) \\ (r_2) & b_{k-2} \quad a \rightharpoonup c \text{—} c \\ (r_3) & \varepsilon_1 \quad a \rightharpoonup \ell_1 \text{—} \ell_{final} \\ (r_4) & \varepsilon_i \quad \ell_i \rightharpoonup c \text{—} \ell_{i+1} \quad 1 \leq i \leq n \\ (r_5) & \varepsilon_m \quad \varepsilon_m \rightharpoonup \varepsilon_{m+1} \text{—} c \quad 1 \leq m \leq (n-1) \end{cases}$$

These rules will produce the leader-asymmetric, single leader crystals of size $p - 1 = k(2)^n$, exactly the same way as in Φ_B with ℓ_{n+1} as a leader. An extra node, ℓ_{final} is then connected to ℓ_{n+1} to give a crystal of size $p = k(2)^n$. In this case, maximum leader to node distance, *d = n+k*

D. Rules for Crystals of Size $p = k(q)^n$. $\quad k, q \geq 3,\ n \geq 0$

Consider the following rule set Φ_D

$$\Phi_D =$$

$$\begin{array}{lll} (r_0) & a \quad a \rightharpoonup \ell_1 \text{—} b_1 & \\ (r_1) & b_i \quad a \rightharpoonup c \text{—} b_{i+1} & 1 \leq i \leq (k-3) \\ (r_2) & b_{k-2} \quad a \rightharpoonup c \text{—} c & \\ (r_3) & \ell_j \quad \ell_j \rightharpoonup \ell_{j+1} \text{—} \varepsilon_{j,1} & 1 \leq j \leq n \\ (r_4) & \ell_j \quad \varepsilon_{j,m} \rightharpoonup \varepsilon_{j,(m+1)} \text{—} c & 1 \leq j \leq n,\ 1 \leq m \leq (q-3) \\ (r_5) & \ell_j \quad \varepsilon_{j,(q-2)} \rightharpoonup c \text{—} c & 1 \leq j \leq n, \end{array}$$

Claim: Φ_D gives leader-asymmetric, single leader crystals of size $p = k(q)^n$.

Proof: Let τ^D be a trajectory produced by Φ_D. Here τ_{k-2} is a path graph with $|\tau_{k-2}| = k$ having a single leader ℓ_1 produced by first three rules r_0, r_1 and r_2. In the next step q of these identical τ_{k-2} graphs are connected via their leaders only, such that these ℓ_1's are themselves connected

in a path graph now with ℓ_2 as their leader to give τ_{k+q-3}. Now by the direct application of Theorem 4.5, τ_{k+q-3} is also leader-asymmetric with ℓ_2 as a leader. Also $\mid \tau_{k+q-3} \mid = kq$. In the next step q identical τ_{k+q-3} are connected via their leaders ℓ_2 that are connected in a path graph, thus giving us $\tau_{(k-2)+2(q-1)}$ with ℓ_3 as a leader and $\mid \tau_{(k-2)+2(q-1)} \mid = q(kq) = k(q)^2$. Again $\tau_{(k-2)+2(q-1)}$ is leader-asymmetric by the direct application of Theorem 4.5. This continues n times until we get a terminal graph $\tau_{(k-2)+n(q-1)}$ which is infact a leader-asymmetric with ℓ_{n+1} as a leader and $\mid \tau_{(k-2)+n(q-1)} \mid = k(q)^n$. ■

Here, maximum leader to node distance, $d = n(q-1)+k-1$.

VII. Example and General Algorithm

An example showing the construction of crystals of size $p = 8$, using the rule sets of Section VI-A are shown in the Fig. 8. All rules in the rule sets in Section VI are binary [2]. An algorithm behind the graph grammars of above cases is presented below.

Algorithm I

Require: $\mathscr{G}$ be a graph, such that
(a) $\mathscr{G}$ is leader-asymmetric with a single leader
(b) $\mid \mathscr{G} \mid = p$

1: **Factorize** p as $p = k(q)^n$
2: **Make Path Grpahs** $\mathscr{G}_1$ initially with a single leader ℓ_1 and $\mid \mathscr{G}_1 \mid = k$
3: **for** $i = 1$ to $i = n$
4: $\mathscr{G}_{i+1}$ = Connect q no. of $\mathscr{G}_i$'s together via their leaders ℓ_i, s.t. these ℓ_i's are connected in a path graph with the end node as a new leader ℓ_{i+1} of $\mathscr{G}_{i+1}$.
5: $i = i+1$
6: **end**
7: $\mathscr{G}_{n+1}$ is required $\mathscr{G}$ with a single leader ℓ_{n+1}

Here the factorization step of $p = k(q)^n$ is important as it is not unique. The maximum leader to node distance d depends on the specific choice of k, q and n for the same p. It turns out that for same $p = k(q)^n$, factorization with a larger value of n produces a crystal of size p with smaller d, where d is the maximum leader to node distance, if we use the above scheme. Also, for same p, if two factorizations have same n, then the one with larger q produces a crystal with smaller d.

VIII. Conclusions

In this paper, we discussed the construction of hierarchical leader-follower networks through the interconnection of multiple subnetworks that are themselves leader-follower.

[2]Rules, whose vertex sets have two vertices are binary.

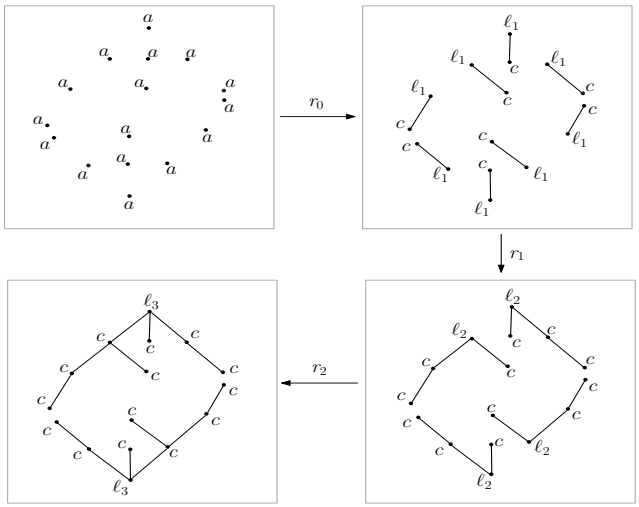

Fig. 8. An example assembly sequences for producing crystals of size $p = 8$

We investigated the leader-asymmetry property of such interconnected networks which is a necessary condition for their controllability, in terms of their subnetworks. We also gave a sufficient condition for the interconnected hierarchical networks to be leader-asymmetric. Moreover, these results are used to design the rules for self-assembly of isolated nodes into leader-asymmetric, single leader networks of any size in a decentralized manner.

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