

## Math 1496 - Sample Test 2

1. Find the derivative of the following

$$(i) \quad y = x \sin^{-1} x$$

$$(ii) \quad y = \ln(x^2 + e^x)$$

$$(iii) \quad x^2 - xy + y^4 = x - y$$

$$\text{Soln (i). } y' = 1 \sin^{-1} x + x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\text{Soln (ii). } y' = \frac{1}{x^2 + e^x} \cdot (2x + e^x)$$

Soln (iii). We differentiate implicitly giving

$$\begin{aligned} 2x - (xy' + y) + 4y^3y' &= 1 - y' \\ -xy' + 4y^3y' + y' &= 1 - 2x + y \\ (-x + 4y^3 + 1)y' &= 1 - 2x + y \\ y' &= \frac{1 - 2x + y}{-x + 4y^3 + 1} \end{aligned} \tag{1}$$

$$y' = \frac{1}{x^2 + e^x} \cdot (2x + e^x)$$

2. If  $f(x) = x^3 + x + 1$  find  $(f^{-1})'(3)$ .

Soln. The inverse is given by  $x = y^3 + y + 1$ . We wish to find the derivative at  $x = 3$  so we need the  $y$  value. Therefore,  $3 = y^3 + y + 1$  or  $y^3 + y - 2 = 0$  or  $(y - 1)(y^2 + y + 2) = 0$  giving that  $y = 1$ . Since  $x = y^3 + y + 1$  then  $1 = (3y^2 + 1)y'$  or  $y' = \frac{1}{3y^2 + 1}$  and at  $y = 1$   $y' = 1/4$ .

3. Find the absolute minimum and maximum of the following on the given interval

$$(i) \quad f(x) = 1 - x^2 \quad \text{on } [-2, 3]$$

$$(ii) \quad f(x) = 2x^3 - 15x^2 + 24x \quad \text{on } [0, 3]$$

Soln (i). Since  $f$  is continuous on  $[-2, 3]$  and differentiable on  $(-2, 3)$  it will have a minimum and maximum. These will be located at the endpoints or inside the interval at the critical points. So here  $f' = -2x$  and  $f' = 0$  when  $x = 0$ .

$$f(-2) = -3, \quad f(0) = 1(\text{max}), \quad f(3) = -8(\text{min}). \tag{2}$$

Soln (ii). Since  $f$  is continuous on  $[0, 3]$  and differentiable on  $(0, 3)$  it will have a minimum and maximum. These will be located at the endpoints or inside the interval at the critical points. So here  $f' = 6x^2 - 30x + 24 = 6(x - 1)(x - 4)$  and  $f' = 0$  when  $x = 1, 4$  but  $x = 4$  is outside the interval

$$f(0) = 0, \quad f(1) = 11(\text{max}), \quad f(3) = -9(\text{min}). \tag{3}$$

4. State the Mean Value Theorem. Verify the Mean Value Theorem for the following

$$(i) \quad f(x) = x^3 - x \quad \text{on } [0, 2]$$

$$(ii) \quad f(x) = \frac{x}{x+2} \quad \text{on } [1, 10]$$

*Soln.* The MVT states that if  $f(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  then there exists a  $c$  (at least one) in  $(a, b)$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(c). \quad (4)$$

*Soln (i)* We have  $\frac{f(2) - f(0)}{2 - 0} = \frac{6}{2} = 3$ . Also  $f' = 3x^2 - 1$  and  $f'(c) = 3c^2 - 1 = 3$  gives  $c = 2/\sqrt{3}$

*Soln (ii)* We have  $\frac{f(10) - f(1)}{10 - 1} = \frac{1/2}{9} = 1/18$ . Also  $f'(x) = \frac{2}{(x+2)^2}$  and

$$f'(c) = \frac{2}{(c+2)^2} = \frac{1}{18} \Rightarrow c = -8, 4$$

from which we choose  $c = 4$ .

5. If  $y = x(x - 4)^3$  calculate the following

(i) The critical numbers

(ii) When  $y$  is increasing and decreasing.

(iii) Determine whether any of the critical numbers are minimum or maximum.

(iv) When  $y$  is concave up and down and determine the points of inflection.

(v) Then sketch the curve.

*Soln*

$$y' = (x - 4)^3 + 3x(x - 4)^2 = 4(x - 1)(x - 4)^2$$

and  $y' = 0$  when  $x = 1, 4$  (Critical numbers)

$$y'' = 4(x - 2)^2 + 8(x - 1)(x - 4) = 12(x - 2)(x - 4)$$

and  $y'' = 0$  when  $x = 2, 4$  (possible PI's.)

$x$		1		2		4	
$x - 1$	-	0	+	+	+	+	+
$(x - 4)^2$	+	+	+	+	+	0	+
$(x - 1)(x - 4)^2$	-	0	+	+	+	0	+
slope	\	—	/	/	/	—	/
$x - 2$	-	-	-	0	+	+	+
$x - 4$	-	-	-	-	-	0	+
$(x - 2)(x - 4)$	+	+	+	0	-	0	+
h/v	∪	∪	∪	PI	∩	PI	∪

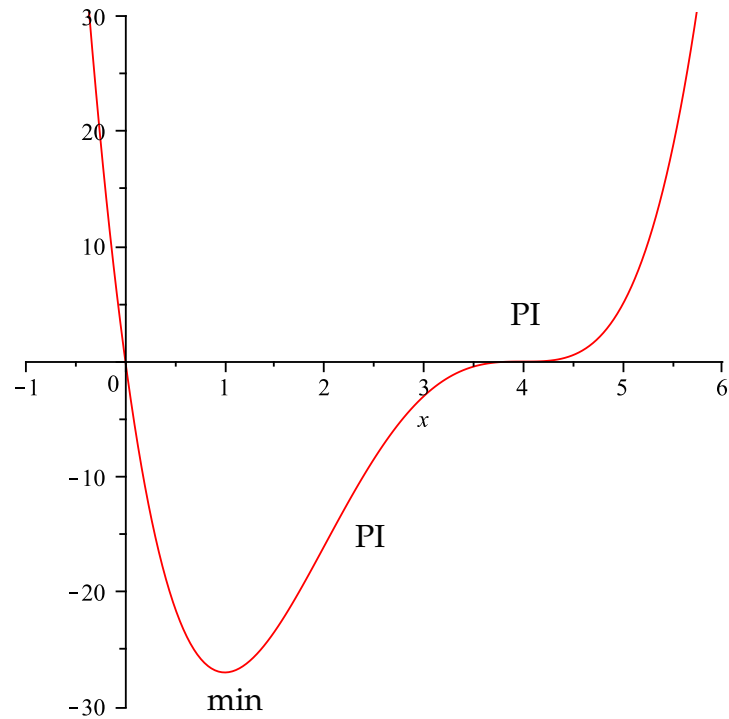
critical numbers  $x = 1, 4$

increasing  $(1, 4), (4, \infty)$  decreasing  $(-\infty, 1)$

min  $(1, -27)$  max - none

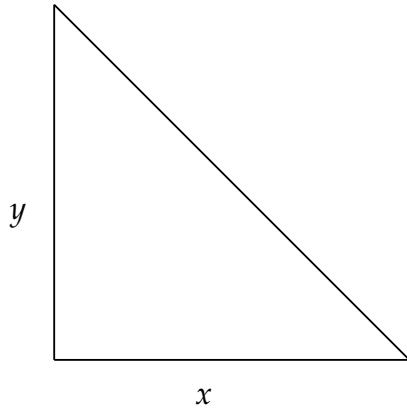
concave up  $(\infty, 2), (4, \infty)$  concave down  $(2, 4)$

PI  $(2, -16), (4, 0)$



6. A ladder 13 feet long is resting against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 ft/sec. At rate is the tip of the ladder moving down the wall when the base of the ladder is 5 ft away from the wall?

*Soln.*



What we know:  $\frac{dx}{dt} = 2$

What we want:  $\frac{dy}{dt}$  when  $x = 5$

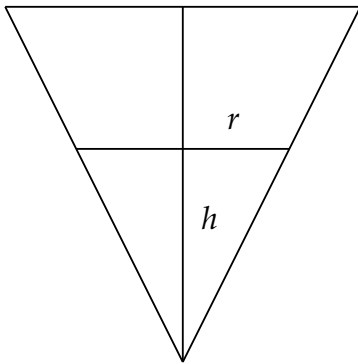
Relate rates:  $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

so  $\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$

When  $x = 4, y = 12$  so  $\frac{dy}{dt} = -\frac{5}{12} \cdot 2 = -\frac{5}{6}$  ft/sec.

7. A paper cup in the shape of an inverted cone with height 10 cm and a base of radius 3 cm, is being filled at a rate of  $2 \text{ cm}^3/\text{min}$ . Find the rate of change in the height of the water when the height of the water is 5 cm.

*Soln.*



What we know:  $\frac{dV}{dt} = 2$

What we want:  $\frac{dh}{dt}$  when  $h = 5$

Relate variables:  $V = \frac{1}{3}\pi r^2 h$ .

We also have similar triangles so  $\frac{h}{10} = \frac{r}{3}$

so  $V = \frac{3\pi h^3}{100}$

Relate rates:  $\frac{dV}{dt} = \frac{9\pi h^2}{100} \frac{dh}{dt} = 2$  so  $\frac{dh}{dt} = \frac{100}{9\pi h^2} \frac{dV}{dt}$

When  $h = 5$   $\frac{dh}{dt} = \frac{100}{9\pi 5^2} \cdot 2 = \frac{8}{9\pi}$  cm/min.