p-adic L-functions for unitary groups

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[1] This work was partially supported by NSF grants DMS-1249384 and DMS-1559609.

Goal for talk

Describe a construction of p-adic L-functions

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- For unitary groups

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- For ordinary families

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- Builds on earlier constructions, including Hida and Katz (and recovers Katz's p-adic L-functions for CM fields as a special case)
- Motivated by various conjectures about existence, form, and role of p-adic L-functions (in Iwasawa Theory), due to Coates, Perrin-Riou, Greenberg, ...

Goals of project

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- More precisely, construct an element $\mathcal{L} \in \Lambda \hat{\otimes} \mathbb{T}$ so that the image of $\mathcal{L}$ under the map induced by a Hecke character χ and a system of Hecke eigenvalues λ_π (from a Hecke algebra $\mathbb{T}$ to an appropriate p-adic ring) is $L(s, \chi, \pi)/\Omega_{\pi,\chi}$ for some period $\Omega_{\pi,\chi}$. (Here Λ is a certain Iwasawa algebra.)

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Remark: By work of Chenevier, there is a family of Galois representations associated to these cuspidal automorphic representations.

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4. p-adic side: Interpret this pairing in terms of a p-adic measure, or equivalently, as an element of $\Lambda\hat{\otimes}\mathbb{T}$.

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2. Overview of interpretation of pairing in p-adic setting, and comparison with automorphic side
3. Explain the construction of a family of Eisenstein series

Doubling method and pullback methods

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- Method for obtaining integral representations of L-function (i.e. whose Euler factors are expressed as certain integrals)
- Good for other classical groups as well
- Unlike the usual Rankin-Selberg method for GL_n or the Langlands-Shahidi method does not rely on Whittaker models

Setup

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- For the discussion of p-adic properties later, we also fix a rational prime p such that each prime in E above p splits in K.

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Note that $\langle , \rangle_W$ is a Hermitian pairing on W of signature (n, n).

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Remark

- Have natural embedding $U(V) \times U(V) \hookrightarrow U(W)$, and identify $U(V) \times U(V)$ with its image in $U(W)$
- Similarly, $G(U(V) \times U(V)) \hookrightarrow GU(W)$

Doubling method: integrals

The doubling method expresses certain L-functions as an integral of a pair of cusp forms on $U(V) \times U(V)$ against an Eisenstein series on $U(W)$

Doubling method input

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- $M \cong \mathrm{GL}_n(K)$

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- Can adapt to include similitude factors, when working with GU instead of U

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- Define Eisenstein series $E_{f_{s,\chi}}$ on U_W by

$$E_{f_{s,\chi}}(h) = \sum_{\gamma \in P(E)\backslash U_W(E)} f_{s,\chi}(\gamma h).$$

Eisenstein series

This Eisenstein series extends to a meromorphic function of s and satisfies a functional equation

Question

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We'll return to that question later.

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Doubling method: Cusp forms

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- Let $\varphi \in \pi$ and $\varphi' \in \pi'$.

Another question

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How should we choose φ and φ'?

Doubling method: integral

Define

$$Z\left(\varphi,\varphi',f_{s,\chi}\right) := \int_{[U_V\times U_V]\backslash[U_V)\times U_V](\mathbf{A}_E)} \varphi(g)\varphi'(h)E_{f_{s,\chi}}\left((g,h)\right)\chi^{-1}\left(\det h\right) dg dh$$

Doubling method: properties

- $Z(\varphi, \varphi', f_{s,\chi})$ can be analytically continued to a meromorphic function of s and satisfies a functional equation.

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- $f_{s,\chi} = \otimes_v f_{s,\chi_v} \in \mathrm{Ind}_{P(\mathbf{A})}^{U_W(\mathbf{A})} (\chi \cdot |\bullet|^s) \cong \otimes' \mathrm{Ind}_{P(E_v)}^{U_W(E_v)} (\chi_v \cdot |\bullet|_v^s)$

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Away from the set S_π of places where π_v is ramified, we choose φ_v, φ'_v to be non-zero unramified vectors such that $\langle \varphi_v, \varphi'_v \rangle_v = 1$, where $\langle , \rangle_v$ is the unique (up to scalar-multiple) invariant pairing

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$$\langle \phi, \phi' \rangle_v = \int_{G_V(E_v) \backslash G_V(\mathbf{A})} \phi(g) \phi'(g) dg.$$

Factorization into Euler product

Then we have the following factorization:

$$Z\left(\varphi, \varphi', f_{s,\chi}\right) = \prod_v Z_v\left(\varphi_v, \varphi'_v, f_{s,\chi_v}\right),$$

where

$$Z_v\left(\varphi_v, \varphi'_v, f_{s,\chi_v}\right) = \int_{U_V(E_v)} f_{s,\chi_v}\left((g,1)\right)\langle \pi_v(g)\varphi_v, \varphi'_v\rangle_v dg$$

for $\Re(s) >> 0$.

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1. The Eisenstein series $E_{f_{s,\chi}}$ can be p-adically interpolated (fits into p-adic measure)
2. We can compute the local integrals in the Euler factors and relate them to familiar L-functions

Local zeta integrals appearing in the Euler product

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Local zeta integrals appearing in the Euler product

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3. $v \nmid p\infty$ such that
   - (unramified case) v is in the set S of primes that does not ramify in K and at which π_v and χ_v are unramified
   - (ramified case) $v \notin S$

Local zeta integrals: $v \nmid p\infty$, unramified case

For φ_v, φ'_v normalized spherical vectors such that $\langle \varphi_v, \varphi'_v \rangle_v = 1$ and $f_{\chi_v,s}$ the unique $G_W(\mathcal{O}_w)$-invariant function such that $f_{\chi_v,s}(K_w) = 1$,

$$d_{n,v}(x, \chi_v)\, Z_v\left(\varphi_v, \varphi'_v, f_{\chi_v,s}\right) = L_v\left(s + \frac{1}{2}, \pi_v, \chi_v\right),$$

with

$$d_{n,v}(s, \chi_v) = \prod_{r=0}^{n-1} L_v\left(2s + n - r, \chi_v|_E \eta_v^r\right),$$

η_v the character on E attached by local CFT to the extension K_w/E_v, w a prime over v, and $L_v(s, \pi_v, \chi_v)$ the standard Langlands Euler factor.

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   computations reduce to those in work of Jacquet and Gelbart-Piatetski-Shapiro-Rallis.
2. v inert:
   computations were completed in work of Li.

Local zeta integrals: $v \nmid p\infty$, ramified case

$$Z_v\left(\varphi_v, \varphi'_v, f_{\chi_v,s}\right) = \mathrm{volume}\left(\mathcal{U}_v\right),$$

where $\mathcal{U}_v$ is an open neighborhood of -1_n contained in the open subset $-1_n \cdot \mathcal{K}_v$, with $\mathcal{K}_v$ an open compact subgroup of G_v that fixes φ_v, and $f_{\chi_v,s}$ is defined in terms of the characteristic function of a closely related lattice.

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Computation of integral is relatively quick and straight-forward. (Carefully making choices takes longer than computing the integral.)

Local zeta integrals: $w|p$

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- The calculations at p are where much of the work lies (both for the integrals and, later in this talk, for the Eisenstein measure).
- Euler factor has similar form to that predicted by Coates
- Since we assume each place v of E above p splits in K, the component of the Hecke character χ at v gives pair $(\chi_{v,1}, \chi_{v,2})$ of characters, and the unitary group is isomorphic to a general linear group.

Local zeta integrals: $v|p$

Euler factor at v is $Z_{1,v} \cdot Z_{2,v}$, where

$$Z_{1,v} = \frac{L\left(s+\frac{1}{2}, \pi_{b_v} \otimes \chi_{2,v}\right)}{\varepsilon\left(s+\frac{1}{2}, \pi_{b_v} \otimes \chi_{2,v}\right) L\left(-s+\frac{1}{2}, \pi'_{b_v} \otimes \chi_{2,v}^{-1}\right)}$$

$$Z_{2,v} = \omega_{a_v}(-1) \frac{L\left(\frac{1}{2}+s, \pi'_{a_v} \otimes \chi_{1,v}^{-1}\right)}{\varepsilon\left(s+\frac{1}{2}, \pi'_{a_v} \otimes \chi_{1,v}^{-1}\right) L\left(-s+\frac{1}{2}, \pi_{a_v} \otimes \chi_{1,v}\right)}.$$

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The representations π_v are defined in terms of representations $\pi_{a_v}, \pi_{b_v}, \pi'_{a_v}, \pi'_{b_v}$ dependent on the signature (a_v, b_v) of the unitary group, and these representations are in turn defined by inducing characters on a torus (more later).

Local zeta integrals: $v|p$

Calculation of the integrals relies in part on realizing the integrals in the form of the "Godement-Jacquet" integrals in Jacquet's Corvallis article.

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- Start with f_v built from canonical automorphy factors (more later)
- Possibly apply differential operator to handle non-holomorphic or non-scalar weight case (more on those later, related to differential operators in de Shalit's talk)
- Take φ_v, φ'_v in the highest weight subspace of the archimedean component

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- When the extreme $\mathcal{K}$-type ($\mathcal{K}$ a maximal compact) is one-dimensional, the archimedean zeta integrals were computed by Garrett and also by Shimura.
- When at least one of the two factors of the extreme $\mathcal{K}$-type is one-dimensional, Garrett computed the integrals precisely.
- In all cases, Garrett has shown the integrals are algebraic up to a predictable power of π.

Plan for talk

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2. Overview of interpretation of pairing in p-adic setting, and comparison with automorphic side
3. Explain the construction of a family of Eisenstein series

About weights

We will choose the cusp forms and Eisenstein series so that their weights are compatible (like in related constructions of p-adic L-functions, including Hida, Panchishkin,...).

Special case: definite case

- Recall:

$$Z\left(\varphi, \varphi', f_{s,\chi}\right) := \int_{[U_V \times U_V]\backslash[U_V) \times U_V](\mathbf{A}_E)} \varphi(g)\varphi'(h) E_{f_{s,\chi}}\left((g,h)\right)\chi^{-1}\left(\det h\right) dgdh$$

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\end{aligned}
$$

- In the definite case, we can reinterpret this integral as a finite sum over CM points. This is essentially the strategy of N. Katz.
- So congruence between cusp forms and congruence between Eisenstein series implies congruence between values of L-functions.

The space $X := X_p$

For each integer $r > 0$, let

$$U_r = (\mathcal{O} \otimes \widehat{\mathbb{Z}}^{\{p\}})^\times \times (1 + p^r \mathcal{O} \otimes \mathbb{Z}_p) \subset (K \otimes \widehat{\mathbb{Z}})^\times$$

and

$$X_p = \varprojlim_r K^\times \backslash (K \otimes \widehat{\mathbb{Z}})^\times / U_r.$$

This is the projective limit of the ray class groups of K of conductor (p^r). In particular, it is a profinite abelian group.

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- Let e denote Hida's ordinary projector.
- We will denote a Hecke algebra acting on $\mathcal{V}$ by $\mathbb{T}$ (without further details).

The Gorenstein hypothesis

The $\mathbb{T}$-module $\mathcal{V}$ is said to satisfy the Gorenstein hypothesis if:

- $\mathbb{T} \cong Hom_R(\mathbb{T}, R)$ as R-modules (where R denotes a sufficiently large p-adic ring here)
- $\mathcal{V}$ is free over $\mathbb{T}$

We will assume $\mathcal{V}$ satisfies the Gorenstein hypothesis.

Multiplicity one hypothesis

- The localization of $\mathbb{T}$ at $\ker \lambda_\pi$ is of rank 1 over R (a sufficiently large p-adic ring).
- π appears with multiplicity 1 in the cuspidal spectrum of the unitary group.

Image of the Eisenstein measure

- Define

$$\mathcal{E}^{\mathrm{ord}} \in \Lambda_X \hat{\otimes} \left(\Lambda_T \otimes e\mathcal{V} \hat{\otimes} e\mathcal{V}\right)^T$$

to be the element such that

$$(\chi \otimes \mu)\left(\mathcal{E}^{\mathrm{ord}}\right) = e\phi_{\mathrm{Eis}}(\chi, \mu).$$

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(By $\chi \otimes \mu$, I mean the function obtained by linearly extending the character $\chi \otimes \mu$ on $X_p \otimes T$.)

Image of the Eisenstein measure

In fact, we can actually consider an element

$$\mathcal{E}^{\mathrm{ord}} = \varprojlim_r \mathcal{E}_r^{\mathrm{ord}} \Lambda_X \hat{\otimes} \varprojlim_r \mathrm{Hom}_T\left(C(T/T_r), (e\mathcal{V}_r \hat{\otimes} e\mathcal{V}_r)^{\mathbb{T}_r}\right)$$

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The reason we can take an element $\mathcal{E}^{\mathrm{ord}}$ in this smaller space is that we are actually going to pair the image of the Eisenstein measure with Hecke eigenforms via the doubling method.

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The reason we can take an element $\mathcal{E}^{\mathrm{ord}}$ in this smaller space is that we are actually going to pair the image of the Eisenstein measure with Hecke eigenforms via the doubling method.
So the image of the Eisenstein measure should be viewed as lying in the dual of the space of these eigenforms.

Image of the Eisenstein measure

- In fact, we have an isomorphism

$$\varprojlim_r \mathrm{Hom}_T\left(C(T/T_r), (e\mathcal{V}_r\hat{\otimes}e\mathcal{V}_r)^{\mathbb{T}_r}\right) \cong \mathbb{T}$$

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- So we may view $\mathcal{E}^{\mathrm{ord}}$ as an element of $\Lambda_X \hat{\otimes} \mathbb{T}$.

Isomorphism with Hecke algebra

We now outline the proof that

$$\varprojlim_r \mathrm{Hom}_T\left(C(T/T_r), (e\mathcal{V}_r \hat{\otimes} e\mathcal{V}_r)^{\mathbb{T}_r}\right) \cong \mathbb{T}.$$

Outline of isomorphism with Hecke algebra

- $\mathrm{Hom}_T\left(C(T/T_r), \left(e\mathcal{V}_r \hat{\otimes} e\mathcal{V}_r\right)^{\mathbb{T}_r}\right) \cong \left(e\mathcal{V}_r \hat{\otimes} e\mathcal{V}_r\right)^{\mathbb{T}_r}$

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- So $(e\mathcal{V}_r\hat{\otimes}e\mathcal{V}_r) = \mathrm{Hom}_{\mathbb{T}_r}(e^-\mathcal{V}_r^*, e\mathcal{V}_r)$
- It is a consequence of the Gorenstein and multiplicity one hypotheses that

$$e\mathcal{V}_r \cong \mathbb{T}_r s_r, \qquad e^-\mathcal{V}_r^* \cong \mathbb{T}_r s_r^*$$

- This gives an isomorphism $\mathrm{Hom}_{\mathbb{T}_r}(e^-\mathcal{V}_r^*, e\mathcal{V}_r) \cong \mathbb{T}_r$

Compatibility

Note that the elements s_r's are chosen compatibly so that $s_r = \mathrm{tr}_{T_{r+1}/T_r} s_{r+1}$ and $s_r^* = \mathrm{tr}_{T_{r+1}/T_r} s_{r+1}^*$.

Putting it together

- So we get

$$\varprojlim_r \mathrm{Hom}_T\left(C(T/T_r), (e\mathcal{V}_r \hat{\otimes} e\mathcal{V}_r)^{\mathbb{T}_r}\right) \cong (e\mathcal{V}_r \hat{\otimes} e\mathcal{V}_r)^{\mathbb{T}_r} \cong \varprojlim_r \mathbb{T}_r \cong \mathbb{T}.$$

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- Hence, we may view $\mathcal{E}^{\mathrm{ord}} = \varprojlim_r \mathcal{E}_r^{\mathrm{ord}}$ as an element of $\Lambda_X\hat{\otimes}\mathbb{T} = \Lambda_X\hat{\otimes}\varprojlim_r \mathbb{T}_r$.

Hecke eigenvalues

Now fix a cuspidal automorphic representation π of U_V, and consider a homomorphism

$$\lambda_\pi : \mathbb{T} \to R.$$

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- So $t \cdot T_\pi = \lambda_\pi T_\pi$ for each $t \in \mathbb{T}_r$.
- Under the isomorphism $\mathbb{T}_r \cong \mathrm{Hom}_{\mathbb{T}_r}(e^- \mathcal{V}_r^*, e\mathcal{V}_r)$, t corresponds to $\eta_t : s_r^* \mapsto ts_r$ for each $t \in \mathbb{T}_r$, and $T_\pi \eta_t = \eta_{t \cdot T_\pi} = \lambda_\pi(t)\eta_{T_\pi}$.

Obtaining the p-adic L-function

A three-step process:

1. Realize $\mathcal{E}^{\mathrm{ord}} = \varprojlim_r \mathcal{E}_r^{\mathrm{ord}}$ as an element of $\Lambda_{X_p} \hat{\otimes} \varprojlim_r \mathbb{T}_r \cong \Lambda_{X_p} \hat{\otimes} \varprojlim_r \mathrm{Hom}_{\mathbb{T}_r}\left(e^{-}\mathcal{V}_r^*, e\mathcal{V}_r\right).$

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2. Evaluate at χ to obtain $\chi(\mathcal{E}^{\mathrm{ord}})$ of $\varprojlim_r \mathrm{Hom}_{\mathbb{T}_r}\left(e^{-}\mathcal{V}_r^*, e\mathcal{V}_r\right)$.

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3. Evaluate λ_π on $\chi(\mathcal{E}^{\mathrm{ord}})$ identified with its image under the isomorphism $\varprojlim_r \mathrm{Hom}_{\mathbb{T}_r}\left(e^- \mathcal{V}_r^*, e\mathcal{V}_r\right) \cong \varprojlim_r \mathbb{T}_r$.

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In the next two slides, we explore Steps 2 and 3 further.

Step 2: Considering the element $\chi\left(\mathcal{E}^{\mathrm{ord}}\right) \in \varprojlim_r \mathrm{Hom}_{\mathbb{T}_r}\left(e^-\mathcal{V}_r^*, e\mathcal{V}_r\right)$

- Recall $\chi\left(\mathcal{E}^{\mathrm{ord}}\right)$ is identified with an element E_χ of $\varprojlim_r \left(e\mathcal{V}_r \hat{\otimes} e\mathcal{V}_r\right)^{\mathbb{T}_r}$
- The corresponding element in $\varprojlim_r \mathrm{Hom}_{\mathbb{T}_r}\left(e^-\mathcal{V}_r^*, e\mathcal{V}_r\right)$ is defined by

$$\varphi \mapsto \langle \varphi, E_\chi \rangle,$$

where this is the pairing from the doubling method, identified with the pairing coming from Serre duality.

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$$\varphi \mapsto \langle \varphi, E_\chi \rangle,$$

  where this is the pairing from the doubling method, identified with the pairing coming from Serre duality.
- For $\varphi_\pi \in \pi$, $\langle \varphi_\pi, E_\chi \rangle = L(\pi, \chi) \cdot \varphi_\pi^\iota$.

Step 3: Evaluating λ_π

- For any element $t \in \mathbb{T}$, and $\varphi' \in \pi'$ a Hecke eigenform and $\varphi \in \pi$,

$$\lambda_\pi(t) = \frac{\langle \varphi, t\varphi' \rangle}{\langle \varphi, \varphi' \rangle},$$

  with $\langle , \rangle$ the unique (up to constant multiple) non-trivial invariant pairing.

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- So $\mathcal{E}^{\mathrm{ord}}(\varphi) = \eta_{T_{\mathcal{E}}^{\mathrm{ord}}}\left(T_\pi s_r^*\right) = \eta_{T_{\mathcal{E}^{\mathrm{ord}}} T_\pi}\left(s_r^*\right) = T_{\mathcal{E}^{\mathrm{ord}}} T_\pi s_r = T_{\mathcal{E}^{\mathrm{ord}}} \varphi'$.

Step 3: Evaluating λ_π on $\chi\left(\mathcal{E}^{\mathrm{ord}}\right)$

So (with subscript π's to remind of membership in π) we have

$$\lambda_\pi\left(\chi\left(\mathcal{E}^{\mathrm{ord}}\right)\right) = \frac{\langle \varphi_\pi, \mathcal{E}^{\mathrm{ord}}(\varphi_\pi)\rangle}{\langle \varphi_\pi, \varphi'_\pi\rangle} = \frac{L(\pi,\chi)}{\Omega},$$

where

$$\Omega = \frac{\langle \varphi_\pi, \varphi'_\pi\rangle}{\langle \varphi_\pi, \varphi^\iota_\pi\rangle}.$$

Remarks/reassurance

- There is a unique (up to scalar multiple) ordinary vector $\phi^{\mathrm{ord}} \in \pi_v^{I_{v,r}}$ ($I_{v,r}$ a mod p^r Iwahori subgroup relative to the Borel in the general unitary group)
- Let $0 \neq \phi \in \pi_v^{I_v}$ with $e \cdot \phi = c_\phi \phi^{\mathrm{ord}}$. Then $\langle \phi, \phi' \rangle = c_\phi \langle \phi^{\mathrm{ord}}, \phi' \rangle \neq 0$.

Plan for talk

1. Overview of automorphic side (pairing of Eisenstein series against pair of cusp forms, via *doubling method*)$\checkmark$
2. Overview of interpretation of pairing in p-adic setting, and comparison with automorphic side$\checkmark$
3. Explain the construction of a family of Eisenstein series

Goal for rest of talk

Describe a construction of a p-adic family of Eisenstein series on unitary groups of signature (n, n).

Remarks about related cases

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- Recover N. Katz's results as a special case
- Methods generalize to case of Siegel modular forms
- Very close to - but not precisely - Shimura's Eisenstein series (modify local data at p...more later in talk)

Main Result

Theorem (E, (J. Reine Angew. Math. '15; Algebra Number Theory '14))

There is a p-adic family of Eisenstein series $\{E_\lambda\}$ *on unitary groups (of signature* (n, n)*) indexed by weights* λ*.*

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- *Fix an ordinary CM point $\underline{A}$. Then (modulo a period), the values $E_\lambda(\underline{A})$ vary p-adic continuously as the weights λ vary p-adic continuously.*

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Remark: Can use these Eisenstein series as a starting point to construct p-adic families of automorphic forms on unitary groups of signature (a, b) for all a, b.

The Main Steps in the construction

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2. Fourier coefficients "nice" (easy to describe) and interpolate nicely
3. convenient for p-adic L-functions computations, other applications, etc. (Beware of requirements of potential applications. Try to make "natural" or "general" choices to make more versatile, in case of unforeseen requirements of applications.)

The Main Steps

Step 2: Compute Fourier coefficients (for holomorphic forms), i.e. determine the q-expansions

The Main Steps

Step 3: Apply certain weight-raising C^∞ (and p-adic) differential operators to obtain C^∞ and (p-adic) automorphic forms (For now, call the operators D_∞ and D_{p-adic})

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Step 5: Interpolate special values of C^∞ Eisenstein series, using

$$(*)D_\infty E(\underline{A}) = (*')D_{p-adic}E(\underline{A})$$

for all ordinary CM points $\underline{A}$. (See [E, Ann. Inst. Fourier 2012].)

The Main Steps

Step 6: Obtain a p-adic family of automorphic forms (over the "Igusa tower", over the ordinary locus).

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- Let $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}$ be a Hecke character with conductor dividing p^∞ (can weaken slightly).
- Choose $f \in Ind_{P(\mathbb{A}_E)}^{G(\mathbb{A}_E)}\left(\chi|\cdot|_K^{-s}\right)$.

(Choose f very carefully!)

Siegel Eisenstein series

Define a Siegel Eisenstein series by

$$E_f(g) = \sum_{\gamma \in P(E)\backslash G(E)} f(\gamma g).$$

(Recall:

- P be a Siegel parabolic in $G := U(W)$. Here, $U(W)$ is the unitary group preserving $\langle , \rangle$.
- $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}$ is a Hecke character with conductor dividing p^∞.
- $f \in Ind_{P(\mathbb{A}_E)}^{G(\mathbb{A}_E)} (\chi | \cdot |_K^{-s})$.)

q-expansions

Theorem (E, (J. Reine Angew. Math. '15; Algebra Number Theory '14))

Let R be an $\mathcal{O}_K$-algebra, $k \geq n$. Let $F : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_n(\mathcal{O}_E \otimes \mathbb{Z}_p) \to R$ be a locally constant function supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times GL_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$ satisfying

$$F\left(ex, \mathbb{N}_{K/E}(e)^{-1}y\right) = \mathbb{N}_{k,\nu}(e)F(x,y).$$

($\mathbb{N}_{k,\nu} := \sigma^{k+2\nu}(\sigma\bar{\sigma})^{-\nu}$.) for all $e \in \mathcal{O}_K^\times$, $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$, and $y \in M_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$.

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($\mathbb{N}_{k,\nu} := \sigma^{k+2\nu}(\sigma\bar{\sigma})^{-\nu}$.) for all $e \in \mathcal{O}_K^{\times}$, $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$, and $y \in M_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$.
Then there exists an algebraic automorphic form $G_{k,\nu,F}$ (on $U(n,n)$) of weight (k,ν) defined over R, whose q-expansion at a cusp $m \in GM_+$ is of the form $\sum_{0<\beta\in L_m} c(\beta)q^{\beta}$, with $c(\beta)$ a finite $\mathbb{Z}$-linear combination of terms of the form $F\left(a, \mathbb{N}_{K/E}(a)^{-1}\beta\right)\mathbb{N}_{k,\nu}\left(a^{-1}\det\beta\right)\mathbb{N}_{K/E}(\det\beta)^{-n}$.

q-expansions

Theorem (E, (J. Reine Angew. Math. '15; Algebra Number Theory '14))

Let R be an $\mathcal{O}_K$-algebra, $k \geq n$. Let $F : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_n(\mathcal{O}_E \otimes \mathbb{Z}_p) \to R$ be a locally constant function supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times GL_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$ satisfying

$$F\left(ex, \mathbb{N}_{K/E}(e)^{-1} y\right) = \mathbb{N}_{k,\nu}(e) F(x,y).$$

($\mathbb{N}_{k,\nu} := \sigma^{k+2\nu}(\sigma\bar{\sigma})^{-\nu}$.) for all $e \in \mathcal{O}_K^\times$, $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$, and $y \in M_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$.
Then there exists an algebraic automorphic form $G_{k,\nu,F}$ (on $U(n,n)$) of weight (k,ν) defined over R, whose q-expansion at a cusp $m \in GM_+$ is of the form $\sum_{0<\beta\in L_m} c(\beta)q^\beta$, with $c(\beta)$ a finite $\mathbb{Z}$-linear combination of terms of the form $F\left(a, \mathbb{N}_{K/E}(a)^{-1}\beta\right)\mathbb{N}_{k,\nu}\left(a^{-1}\det\beta\right)\mathbb{N}_{K/E}(\det\beta)^{-n}$.
When $R = \mathbb{C}$, this is the Fourier expansion (at $s = k/2$) of a C^∞ automorphic form $G_{k,\nu,F}(z,s)$ of weight (k,ν) that is holomorphic at $s = k/2$.

Consequence

p-adically interpolate q-expansion coefficients to construct p-adic families of automorphic forms. (Similar to approach taken by Serre, Katz...)

A p-adic Measure

Theorem (E, 2015 (Crelle), 2014 (Algebra Number Theory))

There is a p-adic measure μ on

$$\mathcal{G} := \left((\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times GL_n(\mathcal{O}_E \otimes \mathbb{Z}_p) \right) / \overline{\mathcal{O}_K^\times}$$

with values in the space of p-adic automorphic forms on $U(n,n)$ defined by

$$\int_{\mathcal{G}} H d\mu = G_{n,0,F}$$

for all continuous functions H on $\mathcal{G}$. Here,

$$F(x,y) := \frac{1}{\sigma\left(x^{-1}\mathbb{N}_{K/E}(x)^n \det y\right)^n} H\left(x, y^{-1}\right)$$

extended by 0 to all of $(\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$.

Locally constant case

Remark

For all locally constant functions H,

$$\int_{\mathcal{G}} H(x,y)\det(\mathbb{N}_{K/E}(x)^{-1}y)^{-d}d\mu(\underline{A}) = (*)G_{n+2d,-2d,F(x,y)}(z,k/2),$$

where $\underline{A}$ is an ordinary, CM abelian variety over R, and $z \in$ corresponds to the CM abelian variety $\underline{A}$ (viewed as an abelian variety over $\mathbb{C}$ by extending scalars).

(Can generalize to vector-weight case.)

Form of measure for this project

Theorem (E, 2015 (Crelle), 2014 (Algebra Number Theory))

There is a p-adic measure ϕ_{Eis} (dependent on the signature of our choice of unitary group) on $X_p \times T$ such that

$$\int_{X_p \times T} \tilde{\chi}\mu d\phi_{\mathrm{Eis}} = E_{\chi,\mu}|_{U\times U}$$

with $E_{\chi,\mu}$ Eisenstein series closely related to those of Shimura (and, when $n = 1$, to those of Katz) for μ finite order (and using differential operators, still related even for μ not of finite order).

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How do we choose the Siegel section $f \in Ind_{P(\mathbb{A}_E)}^{G(\mathbb{A}_E)}\left(\chi|\cdot|_K^{-s}\right)$?

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- Fourier coefficients of E_f factor over v
- In particular, we can isolate data at p

Archimedean Siegel Sections

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\begin{aligned}
f_\infty(g;\chi,s) &:= J_g^{k,\nu}(i1_n)^{-1}\left(J_g(i1_n)\overline{J_g(i1_n)}\right)^{k/2-s}\\
J_g(z) &:= \det(cZ+d)\\
J_g^{k,\nu} &:= J_g(z)^{k+\nu}\det(\bar{c}^t Z+\bar{d})^{-\nu}\\
g &= \begin{pmatrix} a & b\\ c & d\end{pmatrix}
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When $s=k/2$ and $\nu=0$, this gives $\frac{1}{(cZ+d)^k}$, appearing in the familiar Eisenstein series $\sum_{(c,d)\neq(0,0)}\frac{1}{(cZ+d)^k}$.

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Corresponding Fourier cofficient is $(*)\det\beta^{k-n}$.

Siegel sections at primes not dividing $p\infty$

Following Shimura, get Siegel sections $\otimes_{v\nmid p\infty} f_v$ such that whenever the corresponding Fourier coefficient $\prod_{v\nmid p\infty} c(\beta, f_v) \neq 0$,

$$\prod_{v\nmid p\infty} c(\beta, f_v) =$$

$$N_{E/\mathbb{Q}}(\mathfrak{b}\mathcal{O}_E)^{-n^2} \prod_{i=0}^{n-1} L^p\left(2s-i, \chi_E^{-1}\tau^i\right)^{-1} \prod_{v\nmid p\infty} P_{\beta,v}\left(\chi_E(\pi_v)^{-1}|\pi_v|_v^{2s}\right),$$

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7. $L^p(r, \chi_E^{-1}\tau^i) = \prod_{v\nmid p\infty \mathrm{cond}\tau}\left(1 - \chi_v(\pi_v)^{-1}\tau^i(\pi_v)\left|\pi_v\right|_v^r\right)^{-1}.$

Sections at primes not divides $p\infty$

Remark

The sections away from $p\infty$ are built from characteristic functions of lattices (which one can choose to have certain properties corresponding to a choice of ideal $\mathfrak{b}$).

Sections at primes dividing p

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Given $\tilde{F} : M_n(\mathcal{O}_E \otimes \mathbb{Z}_p) \times M_n(\mathcal{O}_E \otimes \mathbb{Z}_p) \to R$ subject to certain simple conditions, there exists a Siegel section $f_{\tilde{F}}$ at p whose local Fourier coefficient is $\tilde{F}(1, {}^t\beta)$.

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The Idea: Use "partial Fourier transforms."

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Built from tuple of characters on minors of a certain $n \times n$ matrix.

C^∞ Differential Operators

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If f is a modular form of weight k, then $\partial_k f := y^{-k}\frac{\partial}{\partial z}\left(y^k f\right)$ is a modular form (function) of weight $k+2$.

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The operators ∂_k generalize to the case of unitary groups.

C^∞ Differential operators

Theorem (Shimura)

If f is a holomorphic automorphic form defined over $\bar{\mathbb{Q}}$ of weight ρ, then $(*) D_\rho^d f(\underline{A})$ is in $\bar{\mathbb{Q}}$ for any CM abelian variety $\underline{A}$ defined over $\bar{\mathbb{Q}}$.

(Can generalize to case of vector-valued automorphic forms)

p-adic Differential operators

Let R be an $\mathcal{O}_K$-algebra together with embeddings $R \hookrightarrow \mathbb{C}$ and $R \hookrightarrow \mathbb{C}_p$. Let f be an automorphic form defined over R. Let $\underline{A}$ be an ordinary CM point defined over R.

Theorem (E, 2012)

There's a p-adic differential operator θ_ρ^d such that

$$\theta_\rho^d f(\underline{A}) \in R.$$

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Theorem ((E, 2012 for automorphic forms on unitary groups), generalizes (Katz, 1978 for Hilbert modular forms))

Furthermore,

$$(*')\theta_\rho^d f(\underline{A}) = (*)D_\rho^d f(\underline{A}).$$

Comparison of action of differential operators: general case

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- Because these are holomorphic forms that agree at CM points and since CM points are Zariski dense, these forms agree everywhere.
- Note that the $eD_\infty f = e\mathrm{hol}(D_\infty f)$ follows from the fact that $U_p D_\infty f = U_p \mathrm{hol}(D_\infty f) + p \cdot (\text{nearly holo terms})$ for any classical form f.

p-adic Differential operators

Key Feature: The p-adic differential operators have a "nice" action on q-expansions.

Four related results

1. Scalar-weight (2015, J. Reine Angew. Math.)

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4. Pullbacks without restrictions on the signature, joint work together with J. Fintzen, E. Mantovan, and I. Varma (submitted)
5. See also related results on differential operators - but in a different direction (p inert, $G = U(2,1)$) - in recent work by E. Goren and E. De Shalit

Thank you

A picture of Kubota L-series

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Figure : Kubota standard L-series (from Shreveport Tractor)

For more information, see `http://www.townlineequipment.com/product-spotlights/kubota-standard-l-series.aspx`