Mathematics in Business and Finance

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Abstract

The following is a compendium of equations used in business and finance. Topics include: summation, compounding, yield or net return, net present value, discrete statistics, variance and standard deviation, covariance and correlation, random variables, portfolios, diversification, two asset portfolios, minimum variance portfolios, balance sheets, income statements, bonds, stocks, weighted average cost of capital, holding period returns, options, certainty equivalent, foreign exchange, economic order quantity, pricing, price reaction functions, and taxes. The notation reflects the preference of the author including the more convenient foreshortened summation signs and may include Excel.

Keywords: NPV, portfolio, diversification, bonds, WACC

Here are many of the mathematical equations frequently used in business and finance. In addition, I also provide links to video tutorials where detailed notation and practical application of these equations are discussed in detail. Furthermore, I also referenced some text books and research papers for additional information.

Summation $_{i=1}\Sigma^{?}x_i$ = SUM(A1,A?) in Excel 2010

Mean Average with N securities, T time periods

$\dot{X} = {}_{i=1}\Sigma^{N}x_i/N$ = AVERAGE(A1:A*N*)

$\mu = {}_{t=1}\Sigma^{T}x_t/T$ Greek notation for whole population

Weighted Average $_{i=1}\Sigma^{N}w_ik_i$ where $_{i=1}\Sigma^{N}w_i = 1$ from $w_i = x_i/{}_{i=1}\Sigma^{N}x_i$

Compounding

With *t* years and *k* as a decimal rate

Future value $FV_t = (1+k)^t$ =(1+k1)^t1

Present value $PV_0 = 1/(1+k)^t$ =1/(1+k1)^t1

Doubling: tk% = 72 annually with *k* as a percent for *t* years

Periodic payment *p*eriods $(1+k/p)^{tp}$ as p approaches ∞ then becomes

e^{kt}, $e = 2.71828 = 1 + {}_{i=1}\Sigma^{\infty}1/i!$ = EXP(1)

Perpetuity $_{t=1}\Sigma^{\infty}1/(1+k)^t = 1/k$ = 1/k1

FV Annuity $_{t=1}\Sigma^{T}(1+k)^t = (1/k)([1+k]^T-1)$ =(1/k1)*((1+k1)^t1-1)

PV Annuity $_{t=1}\Sigma^{T}1/(1+k)^t = (1/k)(1-1/[1+k]^T)$ =(1/k1)*(1-1/(1+k1)^t1)

Leases $_{t=0}\Sigma^{T-1}1/(1+k)^t = (1+k)_{t=1}\Sigma^{T}1/(1+k)^t = 1 + {}_{t=1}\Sigma^{T-1}1/(1+k)^t$

Net *R*eturn or yield (see Bodie, Kane, & Marcus)

$R_{it} = (P_{it} - P_{i,t-1} + D_{it})/P_{i,t-1}$ with *P*rice, *t*ime, and *D*ividend for security i

*N*et *P*resent *V*alue

$NPV = {}_{t=1}\Sigma^{T}R_t/(1+k_1)^t - {}_{t=1}\Sigma^{T}C_t/(1+k_2)^t - I_0$ *R*evenue, *C*ost, *I*nvestment;

Risk adjust discount rate k_1 upward, risk adjust k_2 downward

NPV of a loan with a *F*ace value discounted at its own interest rate *i* is zero;

Consider: $F - F/(1+k) - iF/(1+k) = 0$ if i = k.

Internal Rate of Return (IRR) is the k which sets NPV = 0

Modified IRR is $(W_T/I_0)^{1/T} - 1$ where $W_T = {}_{t=1}\Sigma^T(R_t - C_t)(1+y)^{T-t}$ given a yield y.
Discrete Statistics (see Miller & Wilchern)
Chi Square (χ^2) for discrete data in a M by N row and column matrix with totals equal to $K = {}_{i=1}\Sigma^M{}_{j=1}\Sigma^N x_{ij}$
Row and column totals weighted by ${}_{i=1}\Sigma^M w_i = 1$ and ${}_{j=1}\Sigma^N y_j = 1$
Expected values are Kw_iy_j respectively with MN-1 degrees of freedom
Calculated cumulative (Observed – Expected)^2/Expected values or ${}_{i=1}\Sigma^M{}_{j=1}\Sigma^N(x_{ij} - Kw_iy_j)^2/Kw_iy_j$

*Var*iance & Standard Deviation

$Var_A = {}_{i=1}\Sigma^N(x_{Ai} - \dot{X}_A)^2/(N-1)$ for sample size of N, = VAR(A1:A*N*)
$\sigma_A^2 = {}_{t=1}\Sigma^T(x_{At} - \mu_A)^2/T$ Greek for population size of T, V1 =VARP(A1:A*T*)
Standard deviation $s \equiv Var^{1/2}$ = SQRT(V1)=(V1)^.5 = STDEV(A1:A*T*)
Or $\sigma_A \equiv (\sigma_A^2)^{1/2}$ = STDEVP(A1:A*T*)
Confidence bound = $\sigma/N^{1/2}$

*Cov*ariance & Correlation

$Cov_{AB} = {}_{i=1}\Sigma^N(x_{Ai} - \dot{X}_A)(x_{Bi} - \dot{X}_B)/N$ or $\sigma_{AB} = {}_{t=1}\Sigma^T(x_{At} - \mu_A)(x_{Bt} - \mu_B)/T$
$r_{AB} = Cov_{AB}/(s_As_B)$ or $\rho_{AB} = \sigma_{AB}/(\sigma_A\sigma_B)$ and is $\geq$ -1 and $\leq$ 1
$\beta_i = \sigma_{iM}/\sigma_M^2 = \rho_{iM}\sigma_i/\sigma_M$ is a slope, used in CAPM $\mathcal{E}(R_i) = R_f + \beta_i(R_m - R_f)$, where if $R_i' > \mathcal{E}(R_i)$ one accepts or if $R_i' < \mathcal{E}(R_i)$ one rejects.
Treynor slope $(R_i' - R_f)/\beta_i$ > or < $R_m - R_f$ for *m*arket & risk *f*ree *R*ates
(*Un*)*L*evered beta $\beta_L = \beta_U(1+[1\text{-tax}]\text{Debt/Equity})$

Random (see Miller & Wilchern)

*U*niform 0 to 1 distribution is U(0,1) = RAND(), versus
*N*ormal distribution $\mu = 0$ and $\sigma = 1$, where N(0,1) =
$(-2\ln[\check{r}_1])^{1/2}\sin(2\pi\check{r}_2)$, $\check{r}$ = U(0,1), π =3.14159 = $4_{i=1}\Sigma^\infty(1/[4i\text{-}3] - 1/[4i\text{-}1])$ = PI()
Simulating security prices $P_{t+1} = P_t(1 + N[\mu,\sigma])$ given μ is a function of time t and σ generally a function of $t^{1/2}$

*P*ortfolio (see Bodie, Kane, & Marcus)

$R_P = {}_{i=1}\Sigma^N w_iR_i$ and ${}_{i=1}\Sigma^N w_i = 1$ with w<0 for loans
$\sigma_P^2 = {}_{i=1}\Sigma^N{}_{j=1}\Sigma^N w_iw_j\sigma_{ij}$

Diversification

$\sigma_P^2 = {}_{i=1}\Sigma^N{}_{j=1}\Sigma^N 1/N^2\sigma_{ij}$ equally weighted by w=1/N.
If σ_{ii} or σ_{jj} where i = j then is the variance σ_i^2 or σ_j^2.
Thus for i=j there are N variances and for i≠j N^2 - N covariances.
$\sigma_P^2 = {}_{i=1}\Sigma^N 1/N^2\sigma_i^2 + {}_{i=1}\Sigma^N{}_{j=1}\Sigma^N 1/N^2\sigma_{ij}$ for i=j and i≠j respectively or:
$\sigma_P^2 = N/N^2\sigma_i^2 + N(N\text{-}1)/N^2\sigma_{ij}$ and as $N \to \infty$ the idiosyncratic variance $\to$ 0.

Portfolio with N = 2 (see Bodie, Kane, & Marcus)

$R_P = w_AR_A + w_BR_B$ with $w_A + w_B = 1$ and
$\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{AB}$,
If ρ_{AB}=1 then $\sigma_P^2 = (w_A\sigma_A + w_B\sigma_B)^2$ from
$\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\rho_{AB}\sigma_A\sigma_B$ since $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B$

Minimum variance portfolio

Derives from $w_B = (1-w_A)$ or
$\sigma_P^2 = w_A^2\sigma_A^2 + (1-w_A)^2\sigma_B^2 + 2w_A(1-w_A)\sigma_{AB}$ with the 1st derivative of
$\partial\sigma_P^2/\partial w_A = 2w_A\sigma_A^2 - 2\sigma_B^2 + 2w_A\sigma_B^2 + 2\sigma_{AB} - 4w_A\sigma_{AB}$ set equal to zero gives
$w_A = (\sigma_B^2 - \sigma_{AB})/(\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}) = (\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B)/(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)$

Balance Sheet

A = L + E for *A*ssets, *L*iabilities (*D*ebt), & *E*quity
Return on Assets (ROA) ≡ Π/A = Π/S x S/A with *S*ales and Π as profit
Return on Equity (ROE) ≡ Π/E
ROE = ROA x A/E = Π/E = Π/A x A/E and A/E = D/E + 1 = 1/(1-D/A)

Income Statement

\$ = NIAT + Depr. for *N*et *I*ncome *A*fter *T*ax, *Depr*eciation, and \$ as cash
\$ = (Rev – Cost – Depr)(1- tax) + Depr = (Rev – Cost)(1- tax) + tax(Depr)
EBITDA ≡ EBIT + Depr. from *E*arnings *B*efore *I*nterest & *T*axes

Bonds

Paid with twice a year payments of half the amount of interest
$B = iF/k(1 - 1/[1+k]^M) + F/(1+k)^M$ with *B*ond value, k as discount rate or yield to maturity, *i*nterest (coupon) rate, *F*ace value, and *M*aturity. The discount rate k reflects default risk x from risky *C*ash flow discounted at a riskless discount rate y or: C/(1+k) = C(1-x)/(1+y) thus x = (k – y)/(1+k) or
k = (y + x)/(1-x).

Interest rate risk is the elasticity or proportional change in a bond price divided by proportional change in total yield called *D*uration equaling (ΔB/B)/(Δk/[1+k]). It is also the dollar weighted, present value weighted average maturity or $({}_{t=1}\Sigma^M t\$_t/[1+k]^t)/({}_{t=1}\Sigma^M \$_t/[1+k]^t)$ thus the duration of a single payment (zero coupon) bond D = M; for perpetuities is 1+1/k from $(F/[1+k_2] - F/[1+k_1])/\{(F/[1+k_1] + F/[1+k_2])/2\}/\{(k_1-k_2)/(1+[k_1+k_2]/2)\}$.

Immunization is the avoidance of interest rate risk and may be feasible if dollar weighted asset durations equal the dollar weighted liability durations.

Forward *R*ates from time 0 to m or later to n $(1+{}_0R_n) = (1+{}_0R_m)(1+{}_mR_n)$ or $(N-M){}_mR_n \approx N{}_0R_n - M{}_0R_m$ with the Yield Curve from expected inflation, *real* consumption time preference, liquidity, and preferred maturity durations.

Stocks

Stocks often pay dividends quarterly
If $P_0 = {}_{t=1}\Sigma^\infty D_t/(1+k)^t$ and $D_t = D_0(1+g)^t$ and ${}_{t=1}\Sigma^\infty D_0/(1+z)^t = D_0/z$ then
$P_0 = D_0(1+g)^t/(1+k)^t = D_0/(1+z)^t$. Thus z = (k–g)/(1+g). Therefore
$P_0 = D_0(1+g)/(k-g) = D_1/(k-g) = E_1(1-b)/(k-g) = A_0r(1-b)/(k-br)$ with *P*rice, *D*ividends, k as the discount rate sometimes from CAPM, *E*arnings, b is the firm's retention rate for earnings, the firm's achieved *r*ate of *r*eturn, *g*rowth rate often equal to br, and A_0 as the book value of assets.

In equilibrium if r = k then $P_0 = A_0$. If r is greater than k then P_0 is valued above A_0 and vice versa. Values are further increased with higher dividends and payout ratios and lower growth rates and retention rates if r is less than k and vice versa. A price/earnings ratio $(1-b)/(k-br)$ equals $1/k$ in where $r = k$.

*W*eighted *A*verage *C*ost of *C*apital (see Ross & Westerfield)

$$WACC \equiv k_a = w_d k_d + w_e k_e = w_d i(1 - t) + (1 - w_d)(D/P_0 + g)$$

Holding period return (see Bodie, Kane, & Marcus)

Stocks $R = (D_1[1+g]/[k-g] - D_1/[k-g] + D_1)/(D_1/[k-g])$ which results in $R = k$. For non callable bonds $R = (B_1 - B_0 + iF)/B_0$. Thus

$$R = \{iF/k(1-1/[1+k]^{M-1}) + F/(1+k)^{M-1} - iF/k(1 - 1/[1+k]^{M}) - F/(1+k)^{M} + iF\} / \{iF/k(1 - 1/[1+k]^{M}) + F/(1+k)^{M}\}$$

which results in

$R = [-ik + kk + ik(1+k)^{M}]/[i(1+k)^{M} - i + k]$ with the numerator as a multiple of the denominator of k, therefore $R = k$.

Options (see Black-Scholes model)

A call option buys a security and a put option sells a security with intrinsic valuations of $S - X \geq 0$ for calls and $X - S \geq 0$ for puts given *S*ecurity and e*X*ercise/strike prices. Options have excess premiums as a function of: interest rate r (e^{rt}), volatility (σ), time ($t^{1/2}$), "moneyness" ($S \approx X$), and expectations including ex-dividends on puts. The Black-Scholes call option model:

$$S\ N(\{\ln[S/X]+[r+\sigma^2/2]T\}/[\sigma T^{1/2}]) - Xe^{-rT}\ N(\{\ln[S/X]+[r+\sigma^2/2]T\}/[\sigma T^{1/2}] - \sigma T^{1/2})$$

Certainty Equivalent

$U(V_{CE}) = {}_{i=1}\Sigma^{N} w_i U(V_i)$ where ${}_{i=1}\Sigma^{N} w_i = 1$ for *V*alues usually $U(V) = \log(V)$

Foreign Exchange

Interest rate (vs. purchasing power) parity theory:

$(1 + tR_{\$})(FC/\$_t) = (1 + tR_{FC})(FC/\$_0)$ for *R*ates, *F*oreign *C*urrency, and *t*ime.

Economic Order Quantity

An average inventory is $CQ/2$ with *C*arrying cost and *Q*uantity
Ordering cost is TU/Q with *T*ransaction charge and *U*sage.
Total cost is $K = CQ/2 + TU/Q$ and the 1st derivative is
$\partial K/\partial Q = C/2 - TU/Q^2$ set equal to zero gives
$Q = (2UT/C)^{1/2}$ or $CQ/2 = TU/Q$

Pricing

$Q = K/P^{N} = KP^{-N}$ with *P*rice and *Q*uantity, 1st derivative is $\partial Q/\partial P = -NKP^{-N-1}$
The elasticity of demand is $(\Delta Q/Q)/(\Delta P/P)$ or here
$(\partial Q/Q)/(\partial P/P) = (\partial Q/\partial P)(P/Q) = -NKP^{-N-1}P/(KP^{-N}) = -NP^{-N-1}P/P^{-N} = -N$.
$\Pi = R - C$ where Π is profit, *R*evenue, and *C*osts and thus equals
$\Pi = PQ - VQ - F$ with *V*ariable costs and *F*ixed costs, or
$\Pi = PK/P^{N} - VK/P^{N} - F = KP^{1-N} - VKP^{-N} - F$
Consider the 1st derivative of profits with respect to price:
$\partial\Pi/\partial P = (1-N)KP^{-N} + NVKP^{-N-1} = 0$ thus
$(N-1)K/P^{N} = NVK/P^{N+1}$ thus $(N-1) = NV/P$ or
$P^{*} = NV/(N-1) = V/(1-1/\eta)$ where in economics η is also N
$P(N-1) = NV$, $PN-P = NV$, $PN-NV = P$, $N(P-V) = P$ thus

$N = P/(P-V)$

Price Reaction Functions

$\Pi_A = (P_A - V)Q_A - F$

$P_A + td_A = P_B + td_B = P_B + t(D - d_A)$ with *D*istance d and *t*ransportation cost

$d_A = (P_B - P_A)/2t + D/2$ as the distance to equal consumer cost

$Q_A = kd_A = k(P_B - P_A)/2t + kD/2$ with scaling factor k

$\Pi_A = (P_A - V)(k[P_B - P_A]/2t + kD/2) - F$ which has a 1st derivative of:

$\partial\Pi_A/\partial P_A = kP_B/(2t) - kP_A/t + kD/2 + kV/(2t) = 0$

$P_A = (P_B + tD + V)/2$

Taxes (see Higgins "An After Tax Valuation of Debt Instruments")

For after tax cash flows use after tax discount rates $k = i(1-t)$ given the *t*ax rate. The after tax NPV of a loan at its own after tax discount rate is zero; consider: $F - F/(1+k[1-t]) - i(1-t)F/(1+k[1-t])$ equals $F - F(1 + i[1-t])/(1+k[1-t])$ and if $i = k$ then is zero.

Traditional (tax deductible) or Roth (non tax deductible) 2017 IRAs

(see Internal Revenue Service publication 590 and Higgins "In Defense of the Traditional IRA")

IRA $5,500/yr ($6500 over 50), spouse too--working or not*

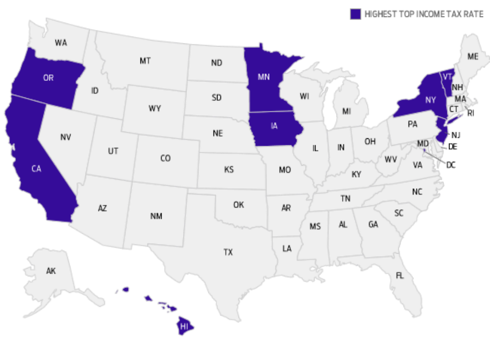
Single to >$62,000 <72,000, Married to >$99,000 <119,000 but if not*

No Traditional IRA benefits arise from qualified dividends or capital gains

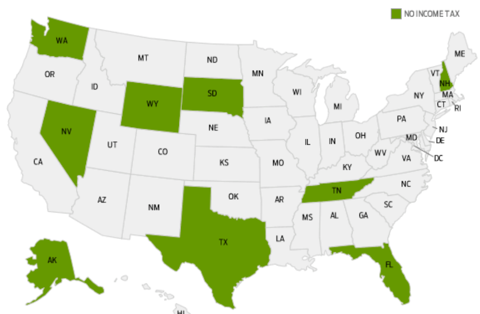
Roth no tax deduction, but gains are tax free:

Single to >$118,000 <$133,000, Married to >$186,000 <$196,000*^

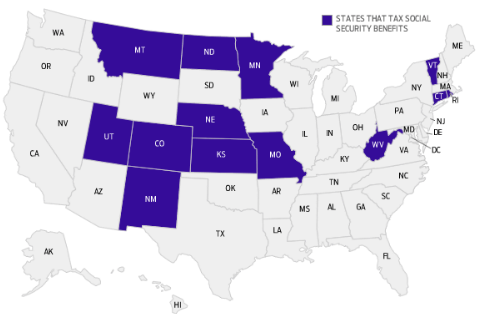
Roth has better return if terminal tax x is larger: $C_t(1-x_t)/C_0(1-x_0)$ vs. C_t/C_0



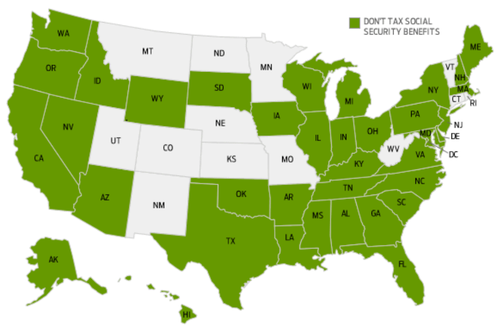
High income tax states



No income tax states



States taxing Social Security



States not taxing Social Security

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Video References

Video discussions, development, and computational examples from my YouTube DrCinvests: https://www.youtube.com/user/drcinvests/videos?sort=dd&view=0&shelf_id=2

Compounding https://www.youtube.com/watch?v=zGdu2DHu6yA https://www.youtube.com/watch?v=o2T6xhxYvPE

Net Present Value https://www.youtube.com/watch?v=B89vwItBFfk

Risk adjust discount rates https://www.youtube.com/watch?v=WXBWO_ee5g0

Variance & Standard Deviation https://www.youtube.com/watch?v=XAhgG-kohtM

Covariance & Correlation https://www.youtube.com/watch?v=ylNwUERjMjY

Diversification https://www.youtube.com/watch?v=fVXWJ18nUkg&t=134s

Minimum variance portfolio https://www.youtube.com/watch?v=fVXWJ18nUkg&t=134s

Balance Sheet https://www.youtube.com/watch?v=_eHllPmGphE&t=111s

Income Statement https://www.youtube.com/watch?v=_eHllPmGphE&t=111s

Bonds https://www.youtube.com/watch?v=E1cY3e6W4Go

Yield Curve https://www.youtube.com/watch?v=oQnsV_eMPwQ

Stocks https://www.youtube.com/watch?v=mjn_pHtsjto

Options https://www.youtube.com/watch?v=HZ88KdCLJVY

Black-Scholes model https://www.youtube.com/watch?v=7R3olJGWF-E

Certainty Equivalent https://www.youtube.com/watch?v=dMQ4EV8LdKU https://www.youtube.com/watch?v=_uxPyZs_trA&t=203s

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Price Reaction Functions https://www.youtube.com/watch?v=zZGiDFRKZGQ

Traditional IRA (tax deductible) https://www.youtube.com/watch?v=Olz1_3H8x3E

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