On the Trade-off Between Controllability and Robustness in Networks of Diffusively Coupled Agents

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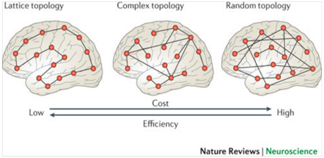
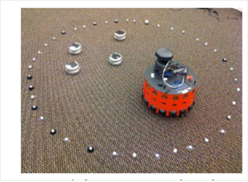
Networked Systems are Everywhere



Natural systems

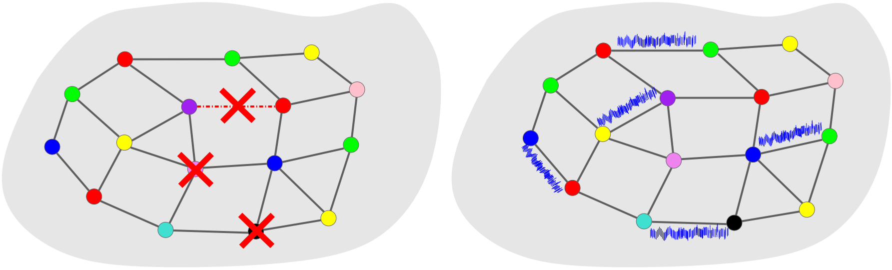


Engineered systems

Controllability

vs

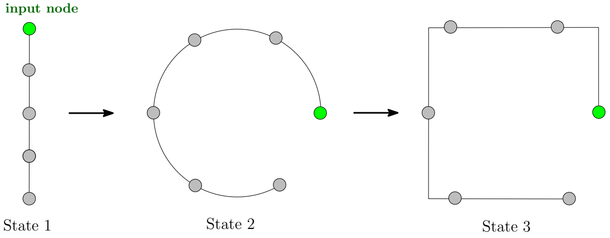
Robustness



Robustness:

- Structural – ability to retain 'structural attributes' in case of node/ link removals.
- Functional – ability to perform 'normally' even in the presence of noise.

Network Controllability – Informal View



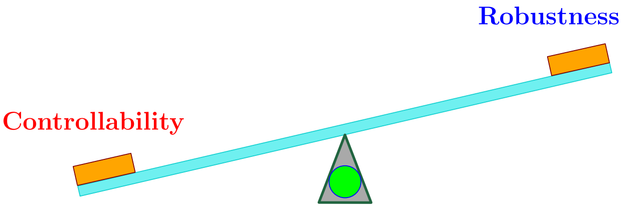
Controllability: Ability to drive the network from a given sate to a desired state by directly manipulating few nodes (external inputs).

Controllability vs Robustness Problem

What are the trade-offs between controllability and robustness in networked dynamical systems?

For a given ‘network parameters’, we are interested in

- maximally controllable networks and their robustness? as well as
- maximally robust networks and their controllability?

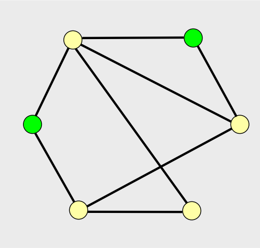


Controllability vs Robustness Problem

What are the trade-offs between controllability and robustness in networked dynamical systems?

Lets set up the problem formally.

- What dynamical networks do we consider?
- How do we measure controllability?
- How do we measure robustness?



Network Dynamics

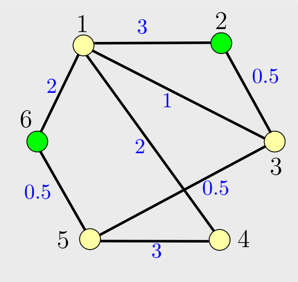
We consider networks of diffusively coupled agents.

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} w_{ij}(x_j - x_i) \qquad \text{(Normal)}$$

$$\dot{x}_\ell = \sum_{j \in \mathcal{N}_i} w_{ij}(x_j - x_i) + u_\ell \qquad \text{(Leaders)}$$

$$\Updownarrow$$

$$\dot{x} = -\mathcal{L}_w x + \mathcal{B}u$$



$\mathcal{L}_w$: weighted graph Laplacian.
$\mathcal{B}$: Input matrix.

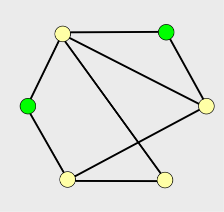
$$\mathcal{L}_w = \begin{bmatrix} 8 & -3 & -1 & -2 & 0 & -2 \\ -3 & 3.5 & -0.5 & 0 & 0 & 0 \\ -1 & -0.5 & 2 & 0 & -0.5 & 0 \\ -2 & 0 & 0 & 5 & -3 & 0 \\ 0 & 0 & -0.5 & -3 & 4 & -0.5 \\ -2 & 0 & 0 & 0 & -0.5 & 2.5 \end{bmatrix}; \qquad \mathcal{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Measuring Control Performance – Strong Structural Controllability

A system is **completely controllable** if there exists an input to drive the system from arbitrary x_{ini} to arbitrary x_{fin}.

- For complete controllability, $\mathbf{\Gamma}$ needs to be **full rank**

$$\Gamma = \begin{bmatrix} \mathcal{B} & -\mathcal{L}_w\mathcal{B} & (-\mathcal{L}_w)^2\mathcal{B} & \cdots & (-\mathcal{L}_w)^{n-1}\mathcal{B} \end{bmatrix}$$



- Rank(Γ) depends on
  - $\mathcal{L}_w$ (edge set and edge weights)
  - $\mathcal{B}$ (choice of leaders)
- For a fixed edge set and leaders,
- **Structural Contr.** – Rank(Γ) with the 'best possible' edge weights.
- **Strong Structural Contr.** – Rank(Γ) with the 'worst' edge weights.

Measuring Control Performance – Strong Structural Controllability

- Strong structural controllability is a stronger and a more general notion of controllability.
- Our measure of controllability is

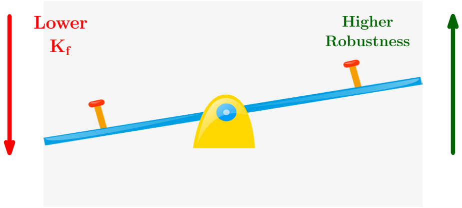
Minimum number of leaders needed to make the network **strong structurally controllable**, that is, Γ is full rank with *any* edge weights.

Measuring Robustness – Kirchhoff Index of Graph

Kirchhoff Index: $K_f = N \sum_i \frac{1}{\lambda_i}$ where, λ_i's are the non-zero eigen values of the (weighted) graph Laplacian.

K_f measures **functional** robust. (expected steady-state dispersion under white noise)

K_f measures **structural** robust. (number and quality of paths between nodes)



Controllability vs Robustness Trade-offs

Lets summarize our setup so far.

Dynamical system:
$\dot{x} = -\mathcal{L}_w x + \mathcal{B} u$

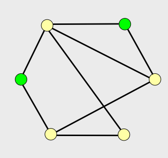
Controllability measure:
Minimum leasers for strong structural controllability

Robustness measure
$\boldsymbol{K_f = N \sum_i \frac{1}{\lambda_i}}$

Objective:
Find extremal networks for these properties for some fixed 'network parameters'.

We consider,

- N – number of nodes
- D – diameter of network



Contributions

Extremal Networks

For *any* N and D,

- Which graphs are maximally robust? What is their controllability performance?
- Which graphs are maximally controllable? What is their robustness?

Strong Structural Controllability Analysis

A graph-theoretic characterization of strong structural controllability in networked systems.

Strong Structural Controllability – A Graph Theoretic View

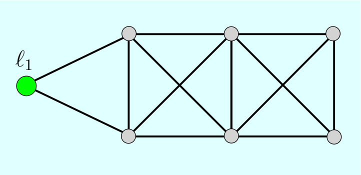
A tight lower bound on the rank of strong structural controllability based on distances between leaders and followers.

Single Leader Case

If ℓ is the leader node, then

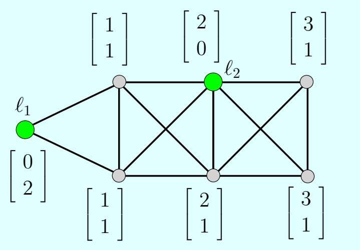
$$\left[\max_{v} \left(\textbf{dist.}(\ell, v)\right) + 1\right] \;\leq\; \textbf{Rank}(\Gamma)$$

- The above bound is sharp.
- With a single leader, Rank(Γ) can always be at least the (diameter + 1) of the network.



Multiple Leader Case –

- First, we define distance-to-leader vector for each node.
- Next, we define a particular sequence of distance-to-leader vectors.



$$\mathcal{S}_A = \left[\begin{bmatrix} \textcircled{0} \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ \textcircled{0} \end{bmatrix}, \begin{bmatrix} \textcircled{1} \\ 1 \end{bmatrix}, \begin{bmatrix} \textcircled{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ \textcircled{1} \end{bmatrix} \right]$$ **valid**

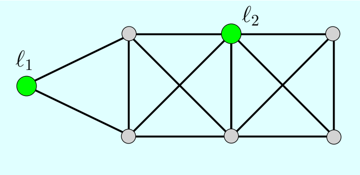
$$\mathcal{S}_B = \left[\begin{bmatrix} \textcircled{0} \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ \textcircled{1} \end{bmatrix}, \begin{bmatrix} 2 \\ \textcircled{0} \end{bmatrix} \right]$$ **not valid**

Multiple Leader Case (TAC 2016)

If γ is the length of the longest valid sequence of distance-to-leader vectors, then

$$\gamma \leq \textbf{Rank}(\boldsymbol{\Gamma})$$

- The above bound is sharp.
- The longest valid sequence can be computed in $O(k(N \log k + N^k))$.
- A greedy heuristic which performs quite well can be implemented in $O(kN^2)$ time.



$$\mathcal{S} = \left[\begin{array}{c} \textcircled{0} \\ 2 \end{array}, \begin{array}{c} 2 \\ \textcircled{0} \end{array}, \begin{array}{c} \textcircled{1} \\ 1 \end{array}, \begin{array}{c} \textcircled{2} \\ 1 \end{array}, \begin{array}{c} 3 \\ \textcircled{1} \end{array}\right]$$

$$\text{Rank}\,(\Gamma) \geq 5$$

Maximally Robust Graphs

Theorem[1] – For any N and D, let G be a graph (weighted or unweighted) with the maximum robustness (minimum K_f) among all graphs with N nodes and diameter D, then G is a *clique-chain* $\mathcal{G}_D(1, n_2, \cdots, n_D, 1)$.

Clique Chains (N nodes and D diameter)

$\mathcal{G}_D(n_1, n_2, \cdots, n_{D+1})$, where $\sum_{i=1}^{D+1} n_i = N$.

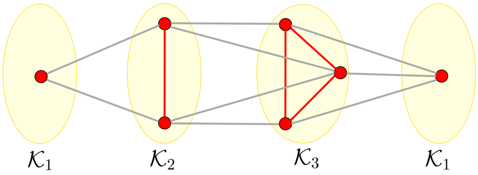


Figure: $N = 7$, $D = 3$, $\mathcal{G}_3(1, 2, 3, 1)$.

[1] W. Ellens, et al. "Effective graph resistance," *Linear Algebra and its Applications*, 2011.

Controllability of Maximally Robust Graphs

Theorem – Let $\mathcal{G}_D(n_1, \cdots, n_{D+1})$ be a clique chain with diameter $D > 2$, and k be the number of leaders needed for the complete strong structural controllability of $\mathcal{G}_D$, then

$$N - (D+1) \;\leq\; k \;\leq N - D. \tag{1}$$

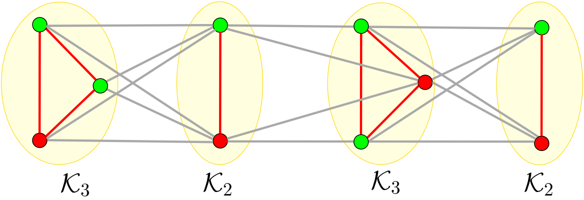


Figure: $\mathcal{G}_3(3, 2, 3, 2)$. Green nodes are leaders (input nodes)

Observation – Maximally robust graphs require a large number of leaders, and hence, perform poorly from controllability perspective.

Maximally Controllable Graphs

- For arbitrary N and D, what do we mean by 'maximally controllable' graphs?
    - Graphs that require 'minimum number of leaders' for complete strong structural controllability.

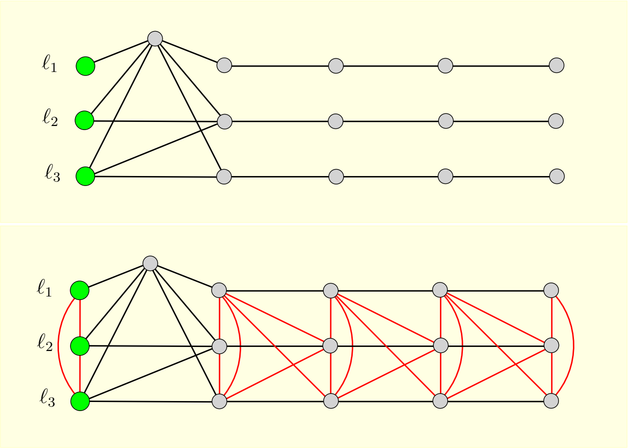
Theorem – For *any* N and D, there exists graphs that have complete strong structural controllability with k leaders, where

$$k \leq \left\lceil \frac{N-1}{D} \right\rceil. \tag{2}$$

- The above bound is sharp, and we cannot do better than that.
- There could be multiple constructions satisfying the above condition.

Maximally Controllable Graphs

- We characterize a couple of such constructions.
- Consider $N = 16$, $D = 5$.
- Minimum leaders required $k = 3$.



Robustness of Maximally Controllable Graphs

Table: K_f of optimal clique chains and maximally controllable graphs $\mathcal{M}$.

N	D	k	$K_f(\mathcal{G}_D^*)$	$K_f(\mathcal{M})$
26	2	13	25.08	35.05
	3	9	28.22	49.36
	4	7	37.63	66.08
	5	5	51.90	107.18
50	2	25	49.04	68.41
	3	17	52.11	95.40
	4	13	64.03	126.22
	5	10	84.31	174.86
100	2	50	99.02	137.77
	3	33	102.05	193.63
	4	25	117.51	252.58
	5	20	148.11	322.26
122	2	61	121.01	168.28
	3	41	124.04	231.81
	4	31	140.68	300.42
	5	25	175.11	376.06

Observation:
Maximally controllable graphs are much less robust as compared to the maximally robust graphs (clique chains) with the same N and D.

We also provide bounds on K_f of maximally controllable graphs.

Conclusions

- Networks that are maximally robust perform poorly from controllability perspective.
- Networks that are maximally controllable exhibit poor robustness.
- A graph-theoretic interpretation of network controllability is crucial in understanding the trade-offs and relationship between network controllability and robustness.

Further Direction: *What are the network operations/modifications that improve one property while minimally deteriorating the other one?*

Thank You