Math 6345 - Adv. ODEs

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If $\dfrac{d\bar{x}}{dt} = A(t)\,\bar{x}$

where $A(t+T) = A(t)$

and $\phi(t)$ the fundemental matrix

then (1) $\phi(t+T) = \phi(t)\,C$ for some const. matrix C

(2) $\phi(t) = P(t)\,e^{Bt}$

where $P(t+T) = P(t)$ B constant matrix

(3) $C = e^{BT}$

(4) under $\bar{x} = P(t)\,\bar{u}$

The original system becomes

$$\dot{\bar{u}} = B\,\bar{u}$$

Ex 1 $\quad \dot{x} = \sin t\, x + y$

$$\dot{y} = (\cos^2 t - \cos t)x - \sin t\, y$$

From the 1st $\quad y = \dot{x} - \sin t\, x$

so the second becomes

$$\ddot{x} - \sin t\cancel{\dot{x}} - \cancel{\cos t x} = (\cos^2 t - \cancel{\cos t})x$$
$$- \sin t(\cancel{\dot{x}} - \sin t\, x)$$

$$\ddot{x} = (\cos^2 t + \sin^2 t)\, x$$
$$= x$$

so $\quad x = c_1 e^{t} + c_2 e^{-t}$

& $\quad y = c_1 e^{t} - c_2 e^{-t} - \sin t(c_1 e^{t} + c_2 e^{-t})$

$$= c_1 (1 - \sin t)e^{t} - c_2(1 + \sin t)e^{t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{t} & e^{-t} \\ (1-\sin t)e^{t} & -(1+\sin t)e^{t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The matrix

$$A(t) = \begin{pmatrix} \sin t & 1 \\ \cos^2 t - \cos t & -\sin t \end{pmatrix}$$

is 2π periodic i.e. $A(t+2\pi) = A(t)$

From the sol^n

$$\Phi(t) = \begin{pmatrix} e^{t} & e^{-t} \\ (1-\sin t)e^{t} & -(1+\sin t)e^{-t} \end{pmatrix}$$

& $$\Phi(t+2\pi) = \begin{pmatrix} e^{t+2\pi} & e^{-t-2\pi} \\ (1-\sin t)\, e^{t+2\pi} & -(1+\sin t)\, e^{-t-2\pi} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t} & e^{-t} \\ (1-\sin t)\, e^{t} & -(1+\sin t)\, e^{-t} \end{pmatrix} \begin{pmatrix} e^{2\pi} & 0 \\ 0 & e^{-2\pi} \end{pmatrix}$$

So $$C = \begin{pmatrix} e^{2\pi} & 0 \\ 0 & e^{-2\pi} \end{pmatrix}$$

(2) we can rewrite $\Phi(t)$ as

$$\Phi(t) = \begin{pmatrix} 1 & 1 \\ 1-\sin t & -(1+\sin t) \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$= P(t)\, e^{Bt}$$

and in this case $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(3) $C = \begin{pmatrix} e^{2\pi} & 0 \\ 0 & e^{-2\pi} \end{pmatrix}$ $\quad e^{Bt} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}$

& $C = e^{2\pi B}$

(4) Under a change of variable

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1-\sin t & -(1+\sin t) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

the system becomes $\dot{\bar{u}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{u}$

to see this let

$$x = u + v$$
$$y = (1-\sin t)u - (1+\sin t)v$$

so $\dot{x} = \sin t\, x + y$

$$\Rightarrow \quad \dot{u} + \dot{v} = \sin t(u+v) + (1-\sin t)u - (1+\sin t)v$$

$$\boxed{\dot{u} + \dot{v} = u - v}$$

and $(1-\sin t)\dot{u} - \cancel{\cos t\, u} - (1+\sin t)\dot{v} - \cos t\, v$

$$= (\cos^2 t - \cos t)(u+v) - \sin t\left((1-\sin t)u - (1+\sin t)v\right)$$

$$= \cos^2 t\, u - \cancel{\cos t\, u} + \cos^2 t\, v - \cos t\, v$$

$$- \sin t\, u + \sin^2 t\, u + \sin t\, v + \sin^2 t\, v$$

$$\boxed{(1-\sin t)\dot{u} - (1+\sin t)\dot{v} = (1-\sin t)u + (1+\sin t)v}$$

and from the 2 boxed equation

$$\dot{u} = u, \quad \dot{v} = -v$$

Ex 2
$$\dot{\bar{x}} = \begin{pmatrix} 1 + \frac{\cos t}{2+\sin t} & 0 \\ 2 & 1 \end{pmatrix} \bar{x}$$

so
$$\dot{x} = \left(1 + \frac{\cos t}{2+\sin t}\right) x \qquad \dot{y} = 2x + y$$

the 1^{st} is separable

$$\frac{dx}{x} = \left(1 + \frac{\cos t}{2+\sin t}\right) dt$$

$$\ln x = t + \ln|2+\sin t| + \ln c_1$$

$$\Rightarrow \quad x = c_1 (2+\sin t) e^{t}$$

$$\dot{y} - y = 2c_1 (2+\sin t) e^{t} \qquad \mu = e^{-t}$$

$$\frac{d}{dt} e^{-t} y = 2c_1 (2+\sin t)$$

$$e^{-t} y = 4c_1 t - 2c_1 \cos t + c_2$$

$$\Rightarrow \quad y = (4t - 2\cos t) e^{t} c_1 + c_2 e^{t}$$

so $\phi(t) = \begin{pmatrix} (2+\sin t)e^{t} & 0 \\ (4t-2\cos t)e^{t} & e^{t} \end{pmatrix}$

Now $A(t) = \begin{pmatrix} 1+\dfrac{\cos t}{2+\sin t} & 0 \\ 2 & 1 \end{pmatrix}$

is 2π periodic

$$\phi(t+2\pi) = \begin{pmatrix} (2+\sin t)e^{t+2\pi} & 0 \\ \left(4(t+2\pi)-2\cos t\right)e^{t+2\pi} & e^{t+2\pi} \end{pmatrix}$$

$$= \begin{pmatrix} (2+\sin t)e^{t} & 0 \\ (4t-2\cos t)e^{t} & e^{t} \end{pmatrix}\begin{pmatrix} e^{2\pi} & 0 \\ 8\pi e^{2\pi} & e^{2\pi} \end{pmatrix}$$

$= \phi(t)\, C$ so $C = \begin{pmatrix} e^{2\pi} & 0 \\ 8\pi e^{2\pi} & e^{2\pi} \end{pmatrix}$

2) $$\Phi(t) = P(t)\, e^{Bt}$$

Not really sure how Φ splits up.

3) $$C = e^{BT} = e^{2\pi B} = \begin{pmatrix} e^{2\pi} & 0 \\ 8\pi e^{2\pi} & e^{2\pi} \end{pmatrix}$$

suggests $$e^{Bt} = \begin{pmatrix} e^{t} & 0 \\ 4te^{t} & e^{t} \end{pmatrix}$$

so $$\begin{pmatrix} (2+\sin t)e^{t} & 0 \\ (4t-2\cos t)e^{t} & e^{t} \end{pmatrix} = \overset{1st}{\begin{pmatrix} & 0 \\ & 1 \end{pmatrix}} \begin{pmatrix} e^{t} & 0 \\ 4te^{t} & e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} 2+\sin t & 0 \\ & 1 \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 4te^{t} & e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} 2+\sin t & 0 \\ -2\cos t & 1 \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 4te^{t} & e^{t} \end{pmatrix}$$

4) $x = (2+\sin t)u$

$y = -2\cos t\, u + v$

Sub $\dot{x} = \left(1 + \frac{\cos t}{2+\sin t}\right) x \qquad \dot{y} = 2x + y$

so $(2+\sin t)\dot{u} + \cos t\, u = \frac{2+\sin t + \cos t}{\cancel{2+\sin t}} \cdot (\cancel{2+\sin t})u$

$(2+\sin t)\dot{u} + \cancel{\cos t\, u} = (2+\sin t)u + \cancel{\cos t\, u}$

$\Rightarrow \dot{u} = u$

$\dot{y} = 2x + y$

$-2\cos t\, \dot{u} + 2\sin t\, u + \dot{v} = 2(2+\sin t)u - 2\cos t\, u + v$

$-2\cos t(\underbrace{\dot{u} - u}_{0}) + \dot{v} = 4u + v$

$\Rightarrow \dot{v} = 4u + v$

$$\begin{pmatrix} u \\ v \end{pmatrix}^{\bullet} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

Finally check

$$e^{\begin{pmatrix}1&0\\4&1\end{pmatrix}t} = \begin{pmatrix} e^t & 0 \\ 4te^t & e^t \end{pmatrix}$$

$$B = \begin{pmatrix}1&0\\4&1\end{pmatrix} \qquad B^2 = \begin{pmatrix}1&0\\4&1\end{pmatrix}\begin{pmatrix}1&0\\4&1\end{pmatrix} = \begin{pmatrix}1&0\\8&1\end{pmatrix}$$

$$B^3 = \begin{pmatrix}1&0\\4&1\end{pmatrix}\begin{pmatrix}1&0\\8&1\end{pmatrix} = \begin{pmatrix}1&0\\12&1\end{pmatrix}$$

$$B^n = \begin{pmatrix}1&0\\4n&1\end{pmatrix}$$

$$e^{Bt} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&0\\4&1\end{pmatrix}t + \begin{pmatrix}1&0\\8&1\end{pmatrix}\frac{t^2}{2!} + \begin{pmatrix}1&0\\12&1\end{pmatrix}\frac{t^3}{3!}$$

$$= \begin{pmatrix} 1+t+\frac{t^2}{2!}+\cdots & 0 \\ 4t\left(1+t+\frac{t^2}{2!}+\cdots\right) & 1+t+\frac{t^2}{2!}+\cdots \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 4te^t & e^t \end{pmatrix}$$

Also $\begin{pmatrix}1&0\\4&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\4&0\end{pmatrix}$

these commute.

$$e^{\begin{pmatrix}1&0\\0&1\end{pmatrix}t} \cdot e^{\begin{pmatrix}0&0\\4&0\end{pmatrix}t} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix}1&0\\4t&1\end{pmatrix}$$