A Theory of Financial Distress in Start-Up Companies

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Abstract

This paper applies results of probabilistic ruin theory to analyze the likelihood that a start-up company will experience financial distress. The intertemporal financial cash inflows to the start-up from the financing source(s) are assumed to be contractual; they are known and certain as to timing and magnitude. The cash outflows of the start-up are assumed to be randomly distributed as to their timing and their magnitude. Moreover, the moral hazard borne by the financing source can be mitigated if the founders of the start-up are required into contribute a minimal equity participation at the inception of business operations. The main theoretical result establishes an equation relating the probability of the start-up's financial distress to three parameters: the parameter characterizing the distribution function governing the cash outflows, the parameter representing the contractual cash inflows from the financing source and the parameter representing the initial equity contribution to the startup.

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I. Introduction

A startup company is an entrepreneurial venture, which is typically a newly emerging, fast-growing business that is intended to meet a marketplace opportunity by developing a sustainable business based on an innovative product, service, process or a platform. A startup is usually a company such as a small business, a partnership or an organization designed to effectively develop and validate a scalable business. Startup businesses are a paradigm of the germination of corporate vitality in the United States. Here are a few statistics to put this into perspective.

In year 2016, 253 venture capital funds raised $451.6 billion, a 10 year high, to deploy to startup companies. More than 7,750 venture-backed companies received $69.1billion in funding in year 2016. [1]

In view of the economic significance of startups, it is surprising that so little attention has been given to the development of a formal theory of financial distress they may experience during the venture capital funding process. There is a large body of literature describing the attitudes and the investment behavior of the venture capitalists.[2] There are empirical studies documenting the realized rates-of-return to the venture capitalists employing different exit strategies.[3] There is an abundant supply of how-to-do-it books and articles dispensing practical guidance to entrepreneurs seeking financing.[4] There are books and articles describing the financial contracting process.[5] However, I have not found any formal analytical theory analyzing the intertemporal riskiness of financing startup companies.

[1] National Venture Capital Association

[2] See, for example, Haar, *et. al* (1988)., Gorman and Sahlman (1989) , Ehrlich *et. al.(1994)* and Mason and Harrison (1995.) For a published description of venture capitalism, written by a well-informed participant in the supply side of the financing process, see Zider (1998.)

[3] See Barry , *et. al.* (1990)

[4] An especially well known how-to-do-it book is Pratt (1993).

[5] A general survey is Hart (2001) A much more focused description of financial contracting in the venture capital industry is Sahlman (1988.)

This paper establishes an analytical theory of the risk of financial distress of startup companies by addressing the properties of two cash flows that characterize a startup: (1) deterministic cash inflows provided by a financing source, usually a venture capital institution and (2) randomly varying cash outflows manifesting the inherent uncertainty of the startup's business environment.

II. Synopsis of the Startup Financing Process

From the perspective of the microeconomic theory of financial institutions, the venture capitalist is a kind of financial intermediary between entrepreneurial startups and the investing public.[6] The venture capitalist manages funds for downstream investors (i.e. the buyers of the stock when, and if, the business goes public) who are not interested in direct investment in high risk / high return investments. Without such financial intermediaries, the market for venture capital would tend to dry up. This is because relatively poorly informed investors who were drawn into failing investments would decline to provide venture capital finance. The investment allocation problems associated with risk and uncertainty of startup companies are assigned to those persons who are willing and competent to manage them. Venture capitalists are sufficiently specialized and experienced in high-risk investments to cope with problems of information asymmetry. This management of information asymmetry tends to lead to the so-called problem of adverse selection. Venture capitalists have been characterized as institutions that resolve information asymmetries.[7]

Venture capital firms often finance startup companies which, at the time of financing, have neither revenues nor even a product in existence.[8] However, a startup company will often approach venture capitalists (hereafter "VCs") for funding at different stages in the gestation of the nascent venture. The multiple stages of the finance solicitation activities have been described by Kozmetsky, *et. al.* (1985)

1. *Seed financing* — capital provided to a startup to prove a concept. It may include product development but does not involve initial marketing
2. *Start-up financing* — financing used in product development and initial marketing.
3. *First-stage financing* — financing provided to startup companies that have expended their initial capital (often in developing a prototype) and require funds to initiate commercial manufacturing and sales.
4. *Second-stage financing* — working capital used for the initial expansion of a startup company that is producing and shipping a product and has growing accounts receivable and perhaps inventories.
5. *Third-stage financing* — funds providing for major expansion of a startup company whose sales volume is increasing and that is breaking even or starting to show a profit.
6. *Fourth-stage, mezzanine, or bridge financing* — capital funds invested in a startup company expected to go public within six months to a year.

When a VC (or a syndicate) makes a funding commitment to a startup company it almost always entails staged financing. That is, the VC makes a firm commitment to offer a fraction of the financing needed, with the understanding that future financing is contingent on firm performance. This is financially equivalent to taking a sequence of call options on the startup business. The VC then evaluates whether the project has reached its performance targets and decides whether to proceed with additional financing. Consequently, the initial funding

[6] See Chan (1983)
[7] See Admati and Pfleiderer (1994) and see Reid (1999.)
[8] Kirilenko (2001.)

commitment is seldom for 100 percent of the entrepreneur's needs. It is at each of those valuation points that the immediate or probable future financial distress will determine whether the cash inflows will continue.

In practice, many startups are initially funded by the founders themselves using "bootstrapping," in which loans or monetary gifts from friends and family are combined with savings and credit card debt to finance the venture. Factoring is another option, though it is not unique to startups. Other funding opportunities include various forms of so-called equity crowd funding, in which the startup seeks funding from a large number of individuals, typically by pitching their idea on the internet.

The financial riskiness of a startup business is manifested in two dimensions: time and money. The theory of these two risks is modeled in Section II.

III. Assumptions Characterizing Cash Flows of a Startup

The start-up's cash flows are characterized by assumptions designed to represent the verisimilitude of the financial risks. These are enumerated below.

(1) The startup's financial liquidity consists of a steady cash flow from VCs or other financing institution(s). The discrete stages of financing described in Section II are generalized to a flow in continuous time line. I assume the average cash inflow from the financing source is a constant and is symbolized by I dollars, measured in units of continuous time. Thus, in the time interval $[\,0, t)$ the startup company receives $I \cdot t$ dollars from the financing source.

(2) At randomly distributed times the startup will experience varying cash outflows: e.g. R&D expenses, technology licensing expenses, ordinary operating expenses, capital expenses, overhead expenses, litigation expenses, etc. At an arbitrary point in time t, the value of the cash outflows at that point in time is symbolized by X_t dollars.

(3) An intrinsic characteristic of any start-up business is the uncertainty respecting the dollar magnitudes of the cash outflows as well as their intertemporal frequency. The first component of the uncertainty is captured by representing the time series of cash outflows $\{X_t \mid t = 1, 2, \ldots\}$ as a stationary stochastic process consisting of i.i.d. random variables governed by a c.d.f. symbolized by $F(x)$. The common expected value of the cash outflows is known and is symbolized by $E(X_1) = E(X_2) = \cdots = E(X)$.

(4) The second manifestation of startup risk is captured by representing the number of cash flow outflows over the time interval [0, t) as a random variable. It is symbolized by $N(t)$. Thus, from the inception of the startup at time $t = 0$, the cumulation of the cash outflows is:

$$Cumulative\ cash\ outflows\ at\ time\ t\ = \sum_{i=1}^{N(t)} X_i \qquad (1)$$

(4) The financing source recognizes the moral hazard in its relationship with the startup as a hidden risk to itself. That hazard can be mitigated (although not eliminated) if the financing institution requires the startup to contribute at least a minimal amount of equity. I assume the start-up's initial capital contribution is a fixed amount, symbolized by K dollars.

The net asset position on the start-up's balance sheet at an arbitrary time t is symbolized by $A(t \mid K, I)$ and can be calculated as:

$$A(t \mid K, I) = K + tI - \sum_{i=1}^{N(t)} X_t \tag{2}$$

IV. Definition of the Start-Up's Financial Distress

Periodically the financing source will examine the financial performance of the startup with a view to deciding whether to renew its call option. As a practical matter, if the examination reveals that the startup is in financial distress, the financing source may refuse to renew the options. That refusal will usually result in the death of the startup.

The cash flow of financing provide by the VCs should be designed to enable the startup to meet its short-term liabilities as they accrue. If the startup cannot liquidate its accounts payable in a reasonable time, it experiences financial distress. The definition of financial distress at an arbitrary point in time is a situation where the net assets on the balance sheet of the start-up are less than or equal to zero. Operationally, this translates to mean a situation where $A(t| K, I) \leq 0$ at time t.

The random character of $A(t|K, I)$ implies that the financial distress can be calculated as the complement of a conditional probability:

$$Prob[A(t \mid K, I) < 0] = 1 - Prob\left[K + tI - \sum_{i=1}^{N(t)} X_i > 0\right] \; for\ all\ t > 0 \tag{3}$$

Equation (3) represents the conditional probability that the start-up will experience financial distress at an arbitrary time t, given the fixed and known values of the start-up's initial equity (K) and the continuous cash flow financing (I). Inasmuch as the financing source will determine the mandatory equity, the conditional probability of financial distress at an arbitrary time can be expressed as a function of that parameter. Hereafter the simplifying notation is adopted:

$$S(K) = Prob\left[K + tI - \sum_{i=1}^{N(t)} X_i > 0, for\ all\ t\right] \tag{4}$$

Expression (4) is the functional form of the probability that the start-up will not experience financial distress, given a fixed initial equity in the amount K and the contractual commitment of cash inflows, I.

V. Definition of the properties of the process $\{N(t)\}$

Equation (4) is an incomplete functional form because the distribution function governing $N(t)$ is not adequately specified. In view of the multiplicity of independent costs associated with each element in the series $\{X_t \mid t = 1, 2, \dots\}$ the simplest approach is to assume that the family of random variables $\{N(t)\}$ depends on the continuous time parameter t such that the increments $N(t_{k+1}) - N(t_k)$ are mutually independent for any finite set $t_1 < t_2 < \cdots < t_n$. This stochastic process has stationary increments if the distribution of $N(s + t) - N(s)$ depends only on the length t of the interval but not on s. Feller [1966,p. 177] has shown that such a process can be represented by a so-called Poisson process with parameter $0 < \theta < 1$:

$$g(t) = \theta e^{-\theta t} \tag{5}$$

Cox [1962, p. 30] has shown that for a Poisson process specified by (5), the number of cash outflows on the time interval $[0, t)$, symbolized by $N(t)$, has a Poisson distribution with mean θt, that is

$$Prob\{N(t) - N(0) = n\} = \frac{(\theta t)^n e^{-\theta t}}{n!} \qquad n = 0, 1, \ldots \qquad (6)$$

VI. Calculation of the Risk of The Startup's Financial Distress

The Poisson process can be applied to calculate the total probability of financial distress conditioned on an arbitrary time for the occurrence of the first Poisson event, symbolized by t_1:

$$S(K) = \int_0^\infty Prob\left[K + It - \sum_{i=1}^{N(t)} X_i > 0, for\ all\ t \mid t = t_1\right] \theta e^{-\theta t}\, dt \quad (7)$$

The conditional probabilistic integrand in Equation (7) can be conditioned again on the magnitude of the first cash outflow, namely X_1:

$$Prob\left[K + It - \sum_{i=1}^{N(t)} X_i > 0, for\ all\ t \mid t = t_1\right]$$
$$= \int_0^\infty Prob\left[K + It - \sum_{i=1}^{N(t)} X_i > 0, for\ all\ t \mid t = t_1, X = X_1\right] dF(x) \qquad (8)$$

The stochastic process $S(K)$ renews itself because after time t_1 the net asset position appearing on the start-up's financial statement is $K + It - x$, given that $t = t_1$ and $x = X_1$ and the stochastic process renews itself with the identical functional form.[9] The defining characteristic expression of a renewal process allows us to write:

$$Prob\left[K + It - \sum_{i=1}^{N(t)} X_i > 0, for\ all\ t \mid t = t_1, X = X_1\right] = S(K + It_1 - X_1) \quad (9)$$

Equations (8) and (9) can be substituted into equation (7) to produce the integral equation:

$$S(K) = \int_0^\infty \left(\int_0^{K+It} S(K + It - x) dF(x)\right) \theta e^{-\theta}\ dt \qquad (10)$$

The Appendix applies the mathematical derivation effected by Karlin and Taylor [1975] to show that equation (10) generates a result showing how the likelihood of financial insolvency is affected if the startup's initial equity increases without limit:

$$\lim_{K \to \infty} S(K) = \frac{S(0)}{1 - \left(\frac{\theta\, E(X)}{I}\right)} \qquad (11)$$

[9] Karlin and Taylor [1975, p 184] state that any integral equation of the form $R(t) = r(t) + \int_0^t R(t - x) dF(x),\ where\ t \leq 0$ is called a renewal equation. In a renewal equation the prescribed (or known) functions are $r(t)$ and the distribution function $F(x)$. The unknown function is $R(t)$.

The product $\theta E(X)$ is the expected cash outflow per unit time. I is the fixed cash inflow per unit time. The limiting result displayed in equation (11) implies a practical result expressed by Proposition 1.

> **Proposition 1:** If a startup company's cash outflows are randomly distributed as a stationary Poisson process and the company's initial equity is close to or equal to zero, the probability that the company will not experience financial distress is dependent exclusively on the ratio of the expected cash outflow per unit time to the fixed cash inflow per unit time.

The function $S(0)$ represents the probability that the start-up will not experience financial distress if its initial equity account is zero. That means all the financing is from a VC or other independent source. Economic logic requires that the limiting value of the function $S(\infty) = 1$. Thus we have the result:

$$S(0) = 1 - \frac{\theta}{I}E(X) \tag{12}$$

Notice that if the expected cash outflow per unit time exceeds the fixed cash inflow, it is certain that the start-up will experience financial distress if the initial equity is zero. That result has a practical economic implication expressed in Proposition 2 below.

> **Proposition 2:** If the initial equity of a startup is zero, its cash outflows are randomly distributed as a stationary Poisson process and the financing source provides fixed periodic cash inflows equal to the expected value of the cash outflows, it is certain that the start-up will experience financial distress.

VII. Concluding Remarks

The practical implications of the two Propositions derived in this paper can inform financing institutions how to allocate infusions of cash to a startup company based on the random character of the cash outflows and the company's initial equity. Generally, the probability of the startup's financial insolvency will not change if cash infusions are constant and are equated to the expected value of the randomly varying cash outflow.

Another practical implication is that the moral hazard reason for requiring an initial equity contribution is supplemented by another rationale: Any increase in the initial equity, *ceteris paribus*, will lower the probability of financial insolvency independently of the randomly varying cash outflows.

APPENDIX

A change of variables $z = K + It$ in the inner integral of equation (11) and rearrangement of it gives

$$S(K)\, e^{-\theta \frac{K}{I}} = \frac{\theta}{I} \int_{Z}^{\infty} \left(\int_{0}^{z} S(z-x) dF(x) \right) e^{-\theta \frac{z}{I}}\, dz \qquad A1$$

The representation A1 in assures that $S(K)$ is differentiable. Differentiation results in

$$e^{-\theta \frac{K}{I}} \left[S'(K) - \frac{\theta}{I} S(K) \right] = -\frac{\theta}{I}\, e^{-\theta \frac{K}{I}} \int_{0}^{K} S(K-x) dF(x) \qquad A2$$

or, equivalently

$$S'(K) = \frac{\theta}{I} \left[S(K) - \int_{0}^{K} S(K-x) dF(x) \right] \qquad A3$$

Integrating both sides of equation A3 with respect to K gives

$$S(w) - S(0) = \frac{\theta}{I} \int_{0}^{w} S(K) dK - \frac{\theta}{I} \int_{0}^{w} \left(\int_{0}^{K} S(K-x)\, dF(x) \right) dK \qquad A3$$

Interchanging the orders of integration and then a change of variable $\varepsilon = K - x$ leads to

$$S(w) = S(0) + \frac{\theta}{I} \int_{0}^{w} S(K) dK - \frac{\theta}{I} \int_{0}^{w} \left(\int_{0}^{w-x} S(\varepsilon) d\varepsilon \right) dF(x) \qquad A4$$

Taylor and Karlin [1975 , p. 211] carry out an integration by parts to show that A4 can be expressed as a renewal equation conforming to the definition of a renewal equation (9) in the text.

$$S(w) = S(0) + \frac{\theta}{I} \int_{0}^{w} S(w-x)[1 - F(x)] dx \qquad A5$$

Expression A5 is a renewal equation with an improper density $\frac{\theta}{I}[1 - F(x)]$ since

$$\int_{0}^{\infty} \frac{\theta}{I} [1 - F(x)]\, dx = \frac{\theta}{I} E(X) \qquad A6$$

By invoking the mathematical properties of a terminating renewal process Taylor and Karlin [1975 ,p. 211] derive the result displayed in equation (11).

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