Leak Detection for Tanks with Hemispherical Bottoms

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Error in Mass Measurement from Non Cylindrical Tank Bottom

Abstract

Mass measurement (also called *hydrostatic tank gaguing*) is used to determine the amount of mass in a storage tank by measuring pressure. This works because $p = \gamma h$ where γ is weight density, and d is the depth of liquid above the pressure sensor p. It has been used for flat bottom tanks for many years.

However, because volume is the basis for transfer of ownership and for billing, mass measurement has not caught on for determining how much petroleum is in a tank. However, mass measurement has has found a niche application for leak detection. In a perfect cyclinder with a flat bottom, the pressure caused by the liquid head is directly proportional to mass even if the volume changes due to thermal expansion.

However, when the bottom is not flat, then significant errors in mass measurement can be made by not taking the tank bottom geometry into account. If the temperature did not change during mass measurement then the accuracy of the mass measurement method would be limited only by the pressure sensing system accuracy. However, temperature is always fluctuating (usually be several ° C). In aboveground tanks the thermal changes are substantial due to the heat transfer from weather. In underground tanks, the main source of thermal variation is from the transfer of fuel into the tanks because ground temperatures tend to be stable. Either way, thermal gradients throughout the liquid exist and must be considered. In a flat bottom cyclindrical tank using mass measurement we show that the temperature effects are canceled by the density changes, but in tanks without flat bottoms they are not.

For the purpose of illustration we consider a cylindrical tank with a hemispherical bottom which are not very common but which do exist in some underground installations in various places in the world. One way to counter the thermal error effect for this type of tank is to wait sufficiently long before a fuel transfer to allow the temperature to decay to a sufficiently constant value - usually several days in larger tanks.

This paper shows why mass measurement accuracy is decreased by the use of a hemispherical bottom as

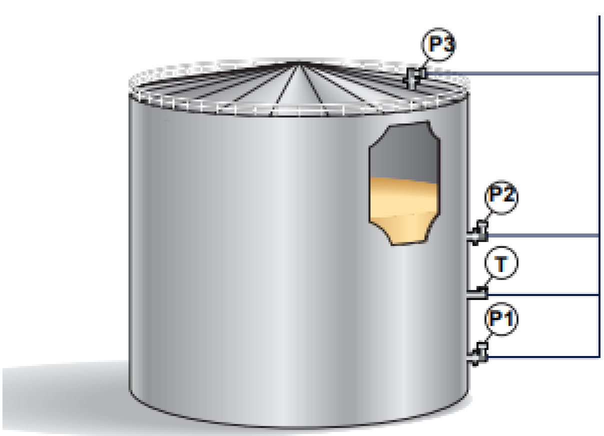

Figure 1: Mass Measurement System Pressure Sensors

temperature changes in the tank. These ideas extend to other shapes for tank bottoms such as conical bottoms as well.

Hydrostatic Tank Guaging Principles

The system used to measure mass is often called a *hybrid tank gauging system* and derives from the fact that it is a combination of a traditional tank gauging system and a Hydrostatic Tank Gauging (HTG) system. There are two main use cases for a hybrid system where the user is interested in either mass or density (or both). Figure 2 shows the measurement concept. Data from P1, P2 and P3 provide the difference in pressures at these point. The difference between P1 and P3 provides the pressure due to the liquid head above P3. If P3 is at or near the bottom then the entire tank mass is measured if density is known. If the fluid, temperature (T) and the pressures are known at P1 and P2, then density can be derived for the particular fluid in the tank. Density together with pressure provide the mass of fluid in the tank. In which operate at or near atmospheric pressure it is often adequate to eliminate the measurement of P3 since pressure gauges subtract off atmospheric pressure anyway. When vapor pressures are significant or the tank settings allow the internal pressure to fluctuate above atmpheric pressure then P3 should be designed to be included in the measurement system.

In a cylindrical flat bottom tank the pressure does not change with thermal expansion because the increase in liquid depth exactly cancels the decrease in density. This is not the case for a tank with other than a flat bottom.

Consider a tank with a hemispherical bottom (see Figure 3). A pressure sensor in the bottom of the tank will yield a systematic error which can be substantial as the temperature in the tank changes.

The rest of this white paper demonstrates and quantifies the potential error caused by temperature changes.

Notation

We consider 2 states:

- Subscript 0 indicates initial conditions or State 0 (lower temperature, higher density)
- Subscript 1 indicates final conditions or State 1 (higher temperature, low density)

The initial condition is the lower temperature. Since temperature is increasing we know that thermal expansion of the contained liquid is taking place.

The variables of interest are V (volume), T (temperature, $^{\circ}C$), γ_0 (weight density), d (depth of liquid), and P pressure measured at bottom of container. A subscript with 1 means the final state (higher temperature). A superscipt A or B refers to the case being considered. If there is no superscript then it applies to both cases. The calculations assume one uniform temperature throughout the liquid container. The container itself is assumed to be at the initial temperature. After the initial calculations, an adjustment for the expansion of the container is made to the relative importance of container expansion.

Thermal Expansion in 3 Dimensions

Thermal expansion can occur in 3 dimensions:

- 1D: $\Delta L = \alpha L_0 \Delta T$
- 2D: $\Delta A = 2\alpha A_0 \Delta T$
- 3D: $\Delta V = 3\alpha V_0 \Delta T$

where α is the linear thermal expansion coefficient, L is the length of a long thin rod, A is a plate shape with length and width of same order of magnitude, and V represent volume.

Liquid thermal expansion (3D) is usually written $\Delta V = \beta V_0 \Delta T$ where $\beta = 3\alpha$.

For jet fuel $\beta \approx 990 \cdot 10^{-6\circ} C$.

For isotropic materials (properties same in all directions) $\alpha_V = \beta = 3\alpha_L$. For small temperature changes the thermal expansion coefficients can be considered constant and this allows us to write

$$V + \Delta V = (L + \Delta L)^3 = L^3 + 3L^2\Delta L + 3L\Delta L^2 + \Delta L^3 \approx L^3 + 3L^2\Delta L = V + 3V\frac{\Delta L}{L} = V(1 + 3\alpha V)$$

Comparing Two Basic Cases

Mass Measurement is the use of hydrostatic pressure at the bottom of a container to account for mass in the container. It has been applied to flat bottom tanks for several decades. The outstanding feature of mass measurement is that the thermal expansion noise in eliminated. But can mass measurement work at the same level of accuracy in a tank without a flat bottom? To understand this unituitive problem we compare a container with a hemispherical shape to a cyclindrical shape shown in Figure 2, both of which have exactly the same volume. In later sections we develop the concepts for containers that have not only hemiperical bottoms but also have cylindrical shapes above the hemisphere as shown in the figures.

Compute Cylindrical Height of Tanks

We compare 2 cases for which the volume is exactly the same:

- Case A: a hemispherical tank completely filled with liquid and
- Case B: a flat bottom vertical cylindrical tank

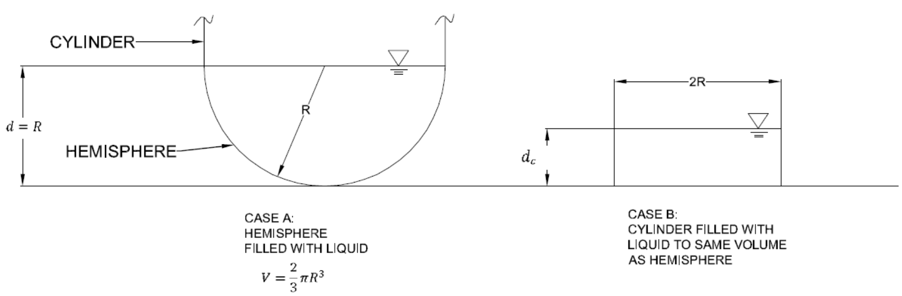

Figure 2: Hemisphere and Cylinder with Same Volume

For Case B the diameter of the tank $2R$ is the same as that of the hemisphere in Case A. The depth of liquid of the cylinder is found by equating volumes for both container A and B to be the same.

$$\begin{aligned} V_0^A =& \frac{2}{3}\pi R^3 = V_0^B \\ V_0^B =& \pi R^2 d_c \\ d_{cyl} =& \frac{2}{3}R \end{aligned}$$

where d_{cyl} is the depth of liquid in a cylinder of radius R containing the same volume as that of the hemisphere.

Compute Volume and Density Change

The volume of both containers A and B has been set equal.

$$V_0^A = V_0^B = \frac{2}{3}\pi R^3$$

The increase in volume for both containers A and B is the same for a given ΔT.

$$\Delta V = \beta V_0 \Delta T$$

For both cases A and and B mass is conserved and $\gamma_0 V_0 = \gamma_1 V_1$ so that the density change in both tanks is the same.

$$\gamma_1 = \gamma_0 \frac{V_0}{V_1} = \frac{\gamma_0 V_0}{V_0 + \beta V_0 \Delta T} = \frac{\gamma_0}{(1 + \beta \Delta T)}$$

Why Mass Measurement Works

Mass measurement works because of the previous equation. If there is a uniform temperature change, the volume change is exactly the reciprocal of the density change. Hence the pressure is constant if there is no mass change in the system in spite of level and volume changes caused by thermal expansion.

$$\begin{aligned}
\frac{V_1}{V_0} &= \frac{1}{1 + \beta \Delta T} \\
\frac{\gamma_0}{\gamma_1} &= 1 + \beta \Delta T \\
\frac{P_1}{P_0} &= \frac{\gamma_0}{\gamma_1}\frac{V_1}{V_0} \\
&= \frac{1 + \beta \Delta T}{1 + \beta \Delta T} = 1.0 \\
P_1 = P_0 &
\end{aligned}$$

But this only applies if the container is cyclindrical with a flat bottom as will be seen.

Compute Change in Liquid Depth

Hydrostatic pressure is given by the equation below; it is the weight density of the liquid multiplied by the depth of liquid at the point of interest below the liquid surface. We can neglect atmospheric pressure since it is additive both above and below the liquid surface.

$$P = \gamma d$$

The pressure sensor in the hemispherical tank A is at the lowest point in the bottom. The pressure sensor in the cylindrical tank B is also at the lowest point on the flat bottom. The thermal expansion of liquid in both cases is contained by the cylindrical portion of the container with radius R that is attached above the hemisphere. Hence, the change in height of liquid due to thermal expanion is exactly the same for Case A and Case B because the expansion is into the cylindrical portion of the container both with radius R.

$$\Delta d = \frac{\Delta V}{\pi R^2} = \frac{V_0 \beta \Delta T}{\pi R^2} = \frac{2}{3}\frac{\pi R^3 \beta \Delta T}{\pi R^2} = \frac{2}{3} R \beta \Delta T$$

Case A Pressure Change

We start with the hemisphere filled with liquid at T_0. The temperature is uniformly increased to T_1 with the expansion of liquid entering the cylindrical portion of the tank. The pressure at state 1 (higher temperature) due to the volumetric thermal expansion is

$$P_1^A = \gamma_1(R + \Delta d) = \gamma_1(R + \frac{2}{3}R\beta\Delta T) = \gamma_1 R(1 + \frac{2}{3}\beta\Delta T) = \gamma_0 R\frac{(1 + \frac{2}{3}\beta\Delta T)}{(1 + \beta\Delta T)}$$

The Case A pressure ratio is

$$Pr^A = P_1^A/P_0^A = \gamma_0 R\frac{(1 + \frac{2}{3}\beta\Delta T)}{(1 + \beta\Delta T)}\frac{1}{\gamma_0 R} = \frac{(1 + \frac{2}{3}\beta\Delta T)}{(1 + \beta\Delta T)}$$

The *pressure ratio* is the ratio of the measured pressure after the temperature change compared to before the temperature change. If the pressure ratio is 1.0 then there is no error in the mass measurement method, but for a tank with a hemispherical bottom an error is introduced as seen above.

The Case A relative pressure change is $rP = \frac{P_1^A - P_0^A}{P_0^A} = Pr - 1$. Then $\Delta P^A = P_0^A \cdot rP = \gamma_1 d_e$ and $d_e = \Delta P^A/\gamma_1$ where d_e is the error in liquid head reported by the pressure measurement. The *relative pressure* shows the change in pressure compared to the initial pressure, which also is a direct measure of the relative error in mass measurement due to thermal changes. d_e is the change in liquid height resulting from the pressure change and it is used to compute the apparent volume change (or reported error in volume).

Case B Pressure Change

The pressure at the bottom of the tank is $P0^B = \frac{2}{3}\gamma_0 R$. The pressure after a change in temperature is

$$P_1^B = \gamma_1(d_c + \Delta d) = \gamma_1(\frac{2}{3}R + \frac{2}{3}\beta R\Delta T) = \frac{2}{3}\gamma_0 R\frac{(1 + \beta\Delta T)}{(1 + \beta\Delta T)} = \frac{2}{3}\gamma_0 R$$

There is of course a volume change from the intital to final temperature of $V(1 + \beta\Delta T)$

The Case B pressure ratio is

$$Pr^B = P_1^B/P_0^B = \frac{\frac{2}{3}\gamma_0 R}{\frac{2}{3}\gamma_0 R} = 1$$

The pressure ratio is 1.0 and therefore the relative pressure change is 0 and so the error caused by thermal changes is zero for mass measurement in a flat bottom cyclindrical tank.

Interpretation

Case A

In case A the pressure ratio is less than one and means that a mass loss is indicated when there really is no loss as the temperature increases and the density decreases. In Case B the pressure ratio is exactly 1.0 so that pressure can be used to measure mass independently of temperature effects.

Some Results

Consider a hemisphere filled with product to depth $d = R$, with radius $R = 50$ and a $1°C$ temperature increase.

```
# Part 1 Hemisphere
R <- 50 #radius of hemisphere
deltaT <- 1 #temperature change deg C
beta <- 990e-6 #thermal expansion coefficent of jet fuel per deg C
gamma0 <- 50 #wt density of jet fuel at lower temperature
gamma1 <- gamma0/(1+beta*deltaT);gamma1 #wt density of jet fuel at higher temperature
```

```
## [1] 49.95055
```

```
V <- (2/3)*pi*R^3;V #volume of hemisphere cf
```

```
## [1] 261799.4
```

```
deltaV <- beta*V*deltaT;deltaV #thermal expansion volume cf
```

```
## [1] 259.1814
```

```
P0 <- gamma0*R;P0 #state 0 pressure at bottom of tank psf
```

```
## [1] 2500
```

```
Pr <- (1+2*beta*deltaT/3)/(1+beta*deltaT); Pr #pressure ratio
```

```
## [1] 0.9996703
```

```
rP <- Pr-1;rP #relative pressure change
```

```
## [1] -0.0003296736
```

```
deltaP <- P0*rP;deltaP #indicated pressure change in psf
```

```
## [1] -0.8241841
```

```
de <- deltaP/gamma1;de #head measurement error in ft
```

```
## [1] -0.0165
```

```
Ve <- de*pi*R^2; Ve #error in cubic ft
```

```
## [1] -129.5907
```

Since the pressure ratio is less than 1.0, the mass measured is under-estimated in the tank with the hemispherical bottom. For Case A the relative pressure change is just about 3 parts per ten thousand as shown for a 1 degree C temperature change. The increase of 1 deg C in a mass measurement system would indicate a loss of $V_e = -129.5907$ cu ft at the higher temperature in spite of there being no change in liquid mass. The actual volumetric expansion is $\Delta V = 259.1814$.

Case B

For Case B the error is zero as the factor $(1 + \beta\Delta T)/(1 + \beta\Delta T) = 1$ and there is no dependence of pressure on ΔT. The relative pressure change from P_0^A is zero. The actual volume change is 259 cubic feet (almost 2000 gallons).

Accounting for the Container Expansion

Because the container dimensions increases on increasing temperature due to the thermal expansion of the steel container, the net effect is to *increase* the error computed above since the apparent liquid contraction is additive with the *increased* container volume. The thermal expansion coefficient of steel may be taken to be about $12 \cdot 10^{-6}$.

```
#Case Container Expansion Error
alpha <- 12e-6; #;linear thermal expansion coefficient of steel
Vs <- (2/3)*pi*(R+alpha*R*deltaT)^3;Vs #thermally expanded hemisphere volume cf
```

```
## [1] 261808.8
```

```
Vs-V #container growth volume
```

```
## [1] 9.424891
```

```
(Vs-V)/deltaV #thermal growth volume for container compared to total thermal expansion
```

```
## [1] 0.03636407
```

The effect of container expansion is less than 4% and will therefore be ignored from here out.

Thermal Expansion in the Cylindrical Part

We now consider the thermal expansion effects when the liquid is above the hemisphere and within the cylindrical part of the container as shown in Figure 3, again comparing the hemispherical bottom to the flat bottom tank. This time the levels (not volumes) are equal for comparison.

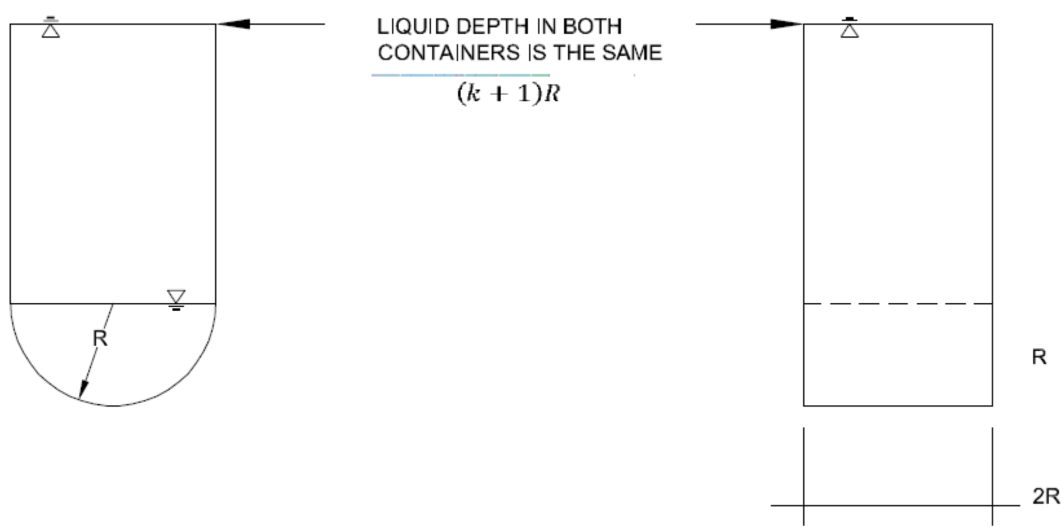

Figure 3: Variable Cylinder Height Comparison

When liquid is in the cylindrical portion of the tank (Figure 3) we can scale the depth of liquid d by the ratio R and we compare two tanks with liquid at the same depth d to the pressure sensors at the bottom of the tanks:

- Case A has a hemispherical bottom

- Case B has the same overall height but is a cylindrical flat bottom container

The initial volume of containers A and B are:

$$
\begin{aligned}
V_0^A \neq& V_0^B \\
V_0^A =& \frac{1}{2}\frac{4}{3}\pi R^3 + \pi R^2 kR \\
=& (k + \frac{2}{3})\pi R^3 \\
V_0^B =& (k+1)\pi R^3
\end{aligned}
$$

The change in container A due to ΔT is:

$$
\begin{aligned}
\Delta V^A =& V_0^A \beta \Delta T \\
=& (k + \frac{2}{3})\pi R^3 \beta \Delta T
\end{aligned}
$$

The change in container B due to ΔT is:

$$
\begin{aligned}
\Delta V^B =& V_0^B \beta \Delta T \\
=& (k+1)\pi R^3 \beta \Delta T
\end{aligned}
$$

Conservation of mass requires that:

$$\gamma_1 = \gamma_0 \frac{1}{1 + \beta \Delta T}$$

Pressure Change in Container A

The change in liquid height in container A resulting from ΔT is:

$$\Delta d^A = \frac{\Delta V^A}{\pi R^2} = \frac{(k + \frac{2}{3})\pi R^3 \beta \Delta T}{\pi R^2} = (k + \frac{2}{3}) R \beta \Delta T$$

The total liquid depth in container A resulting from ΔT is:

$$d^A + \Delta d^A = (k+1)R + (k + \frac{2}{3}) R \beta \Delta T = R[(k+1) + (k + \frac{2}{3})\beta \Delta T]$$

The pressure at the bottom becomes:

$$P_1^A = \frac{\gamma_0}{1 + \beta \Delta T} R[(k+1) + (k + \frac{2}{3})\beta \Delta T]$$

The pressure is dependent on ΔT and hence introduces error into the mass measurement method.

Pressure Change in Container B

For container B we have the change in liquid height resulting from ΔT:

$$\Delta d^B = \frac{\Delta V^B}{\pi R^2} = \frac{(k+1)\pi R^3 \beta \Delta T}{\pi R^2} = (k+1) R \beta \Delta T$$

The total liquid depth in container A resulting from ΔT is:

$$d^B + \Delta d^B = (k+1)R + (k+1) R \beta \Delta T = R(k+1)(1 + \beta \Delta T)$$

The pressure at the bottom becomes:

$$P_1^B = \gamma_0 \frac{1}{1+\beta\Delta T} R(k+1)(1+\beta\Delta T) = \gamma_0 R(k+1)$$

This is the same pressure as the initial pressure and the formula is independent of ΔT. Therefore, no error is introduced by using pressure as a proxy for mass measurement in Case B.

Some Results

Varying the liquid depth d in the cylindrical portion of the tank subjected to the uniform temperature change can provide additional insights about the mass measurement method. In the calculations below we vary the liquid height from a minimum equal to the liquid depth filling the hemisphere (i.e. $d = 50$) to $k = 3$ which represents a total liquid depth of 200 feet ($k + 1 = 4R = 200$).

```
#Part 2 generalized problem covering cyclindrical part of container
k <- seq(0,3,by=.2) #factor for different cyclindrical liquid levels
d <- (k+1)*R #incremental depth
Pr <- ((k+1)+(k+2/3)*beta*deltaT)/((k+1)*(1+beta*deltaT)) #pressure ratio
rP <- Pr-1 #relative pressure ratio
P0 <- gamma0*d #pressure at tank bottom
de <- P0*rP/gamma1 # error in depth
Ve <- pi*R^2*de # error in volume
Vd <- (k+2/3)*pi*R^3 #vol as function of height d
rVe <- Ve/Vd #relative volumetric error
#for combined plot the data needed
d2 <- d #depth in cylindrical part
rVe2 <- rVe #relative volumetric error in cyclindrical part
#PB1 <- gamma0*R*(k+1)
cbind(k_ratio=k,d = d,Pr=Pr,rP=rP,VolErr = Ve, RelVolErr=rVe)
```

```
##       k_ratio   d        Pr            rP    VolErr     RelVolErr
##  [1,]     0.0  50 0.9996703 -3.296736e-04 -129.5907 -4.950000e-04
##  [2,]     0.2  60 0.9997253 -2.747280e-04 -129.5907 -3.807692e-04
##  [3,]     0.4  70 0.9997645 -2.354812e-04 -129.5907 -3.093750e-04
##  [4,]     0.6  80 0.9997940 -2.060460e-04 -129.5907 -2.605263e-04
##  [5,]     0.8  90 0.9998168 -1.831520e-04 -129.5907 -2.250000e-04
##  [6,]     1.0 100 0.9998352 -1.648368e-04 -129.5907 -1.980000e-04
##  [7,]     1.2 110 0.9998501 -1.498516e-04 -129.5907 -1.767857e-04
##  [8,]     1.4 120 0.9998626 -1.373640e-04 -129.5907 -1.596774e-04
##  [9,]     1.6 130 0.9998732 -1.267975e-04 -129.5907 -1.455882e-04
## [10,]     1.8 140 0.9998823 -1.177406e-04 -129.5907 -1.337838e-04
## [11,]     2.0 150 0.9998901 -1.098912e-04 -129.5907 -1.237500e-04
## [12,]     2.2 160 0.9998970 -1.030230e-04 -129.5907 -1.151163e-04
## [13,]     2.4 170 0.9999030 -9.696283e-05 -129.5907 -1.076087e-04
## [14,]     2.6 180 0.9999084 -9.157601e-05 -129.5907 -1.010204e-04
## [15,]     2.8 190 0.9999132 -8.675622e-05 -129.5907 -9.519231e-05
## [16,]     3.0 200 0.9999176 -8.241841e-05 -129.5907 -9.000000e-05
```

In the table:

- k_ratio is the value of k which determines the cylindrical height of the tank relative to the hemisperical height which is R
- d is the liquid depth measured from the bottom

- Pr is the pressure ratio
- rP is the relative pressure ratio
- VolErr is the total volumetric error indicated by the mass measurement error

RelVolErr is the VolErr divided by the volume at different liquid depths

Note that the VolErr column in the table is constant. This means that thermal changes in the cylindrical portion of the tank do not affect the mass measurement error. That is, once the liquid level is beyond the depth of the hemisphere, the error remains constant only affected by changes in the temperature in the hemispherical portion of the container. Of course, the RelVolErr does get smaller as the liquid height in the cyclindrical portion increases. Better relative accuracy is possible when the liquid level is highest. The thermally induced volumetric error remains constant at -129.6 cubic ft as long as the liquid level is in the cylindrical portion of the container. Thus, the relative volumetric error $RelVolErr$ approaches zero in the limit as $d \to \infty$ as seen in Figure 4.

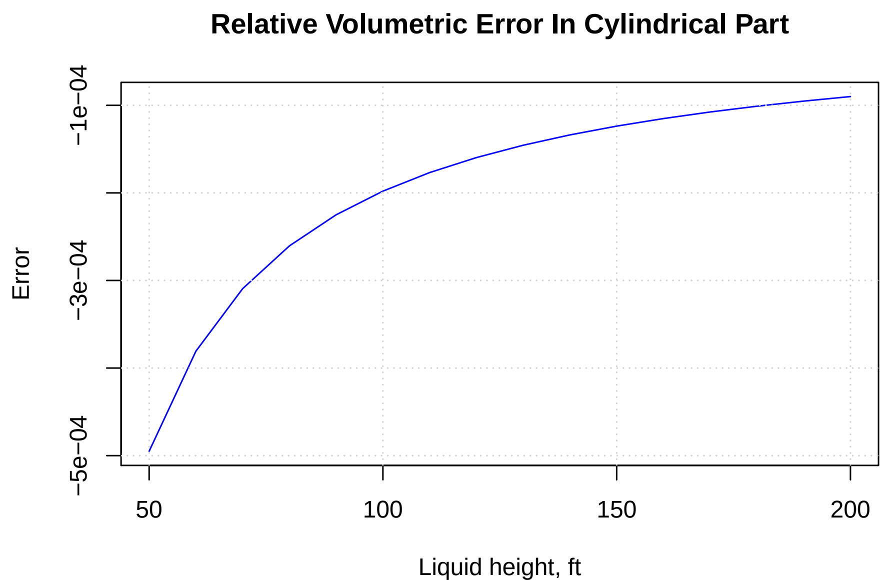

Figure 4: Error in the Cylindrical Portion of Container with Hemispherical Bottom

Liquid Depth Changes in the Hemispherical Section

Here we review the effects of thermal expansion using mass measurement in the hemispherical section of the container where $d < R$. The volume of a *hemispherical cap* is

$$V = \frac{\pi d^2 (3 * R - d)}{3}$$

The hemisphere radius R is related to the depth d and liquid surface radius a by $R = \frac{a^2+d^2}{2d}$ from which the area of the liquid surface at depth d is $A = \pi a^2 = \pi(2R - d^2)$. The change in volume at depth is

$$\Delta V = \frac{\pi d^2 (3R - d)\beta \Delta T}{3}$$

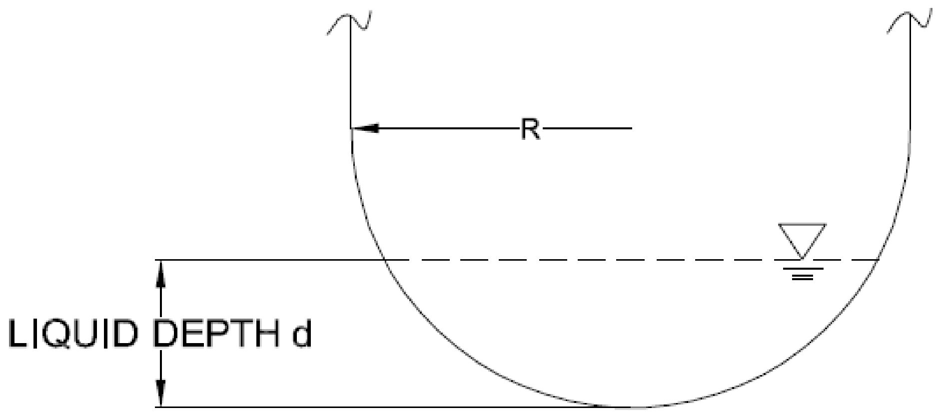

Figure 5: Variable Depth in the Hemispherical Portion

The incremental change in depth is then

$$\Delta d = \frac{\Delta V}{A} = \frac{\pi d^2(3R-d)\beta\Delta T}{3\pi(2Rd-d^2)} = \frac{d^2(3R-d)\beta\Delta T}{3(2Rd-d^2)}$$

The pressure change with depth is given by

$$P_1 = \gamma_1(d+\Delta d) = \frac{\gamma_0}{1+\beta\Delta T}\left(d + \frac{d^2(3R-d)(\beta\Delta T)}{3(2Rd-d^2)}\right)$$

$$\frac{P_1}{P_0} = \frac{P_1}{\gamma_0 d} = \frac{\gamma_0}{(1+\beta\Delta T)\gamma_0 d}\left(d + \frac{d^2(3R-d)(\beta\Delta T)}{3(2Rd-d^2)}\right)$$

which is rewritten

$$Pr = \frac{1}{1+\beta\Delta T} + \frac{d(3R-d)\beta\Delta T}{3(1+\beta\Delta T)(2Rd-d^2} = \frac{3(2Rd-d^2)+d(3R-d)\beta\Delta T}{2(1+\beta\Delta T)(2Rd-d^2)}$$

We have that $rP = 1 - Pr = \frac{\Delta P}{P_0}$.

The error in head is $d_e = \frac{\Delta P}{\gamma_1}$

Behavior in Hemisphere

Since all the thermally induced error for a mass measurement system occurs in the hemisphere it is useful to consider this portion of the container.

```
#part 3
#hemispherical portion
#input data
# R <- 50 #radius of hemisphere
# deltaT <- 1 #temperature change deg C
# beta <- 990e-6 #thermal expansion coeffecent of jet fuel per deg C
# gamma0 <- 50 #wt density of jet fuel at lower temperature
#end input data - can probably be deleted
#gamma1 <- gamma0/(1+beta*deltaT) #wt density of jet fuel at higher temperature
```

```r
d <- seq(from=1,to=50,by=1) #depth of liquid
V <- pi*d^2*(3*R-d)/3 #volume at depth d (based on formula for spherical cap)
deltaV <- V*beta*deltaT #thermal volume change for given volume in cap
A <- pi*(2*R*d-d^2) #surface area of liquid at depth d in cap
deltad <- deltaV/A #liquid level change rate with depth
Pr <- (3*(2*R*d-d^2)+d*(3*R-d)*beta*deltaT)/(3*(1+beta*deltaT)*(2*R*d-d^2)) #pressure ratio
rP <- Pr-1 #relative pressure ratio
P0 <- gamma0*d #pressure at T0 as d
deltaP <- P0*rP #change in pressure with depth
de <- deltaP/gamma1 #pressure head error at depth d
Ve <- de*A #volumetric error at depth d
rVe1 <- Ve/V  #relative volumetric error

cbind(depth=d,deltaV=round(deltaV,4),Pr=round(Pr,8),rP=round(rP,8),Ve=round(Ve,5),rVe=rVe1)
```

```
##       depth  deltaV        Pr          rP         Ve           rVe
##  [1,]     1   0.1545 0.9995072 -0.00049285   -0.15344 -0.0009833557
##  [2,]     2   0.6137 0.9995089 -0.00049115   -0.60545 -0.0009766216
##  [3,]     3   1.3716 0.9995106 -0.00048941   -1.34360 -0.0009697959
##  [4,]     4   2.4218 0.9995124 -0.00048764   -2.35544 -0.0009628767
##  [5,]     5   3.7581 0.9995142 -0.00048583   -3.62854 -0.0009558621
##  [6,]     6   5.3744 0.9995160 -0.00048399   -5.15045 -0.0009487500
##  [7,]     7   7.2643 0.9995179 -0.00048210   -6.90874 -0.0009415385
##  [8,]     8   9.4218 0.9995198 -0.00048018   -8.89096 -0.0009342254
##  [9,]     9  11.8404 0.9995218 -0.00047821  -11.08467 -0.0009268085
## [10,]    10  14.5142 0.9995238 -0.00047620  -13.47743 -0.0009192857
## [11,]    11  17.4367 0.9995259 -0.00047414  -16.05681 -0.0009116547
## [12,]    12  20.6018 0.9995280 -0.00047203  -18.81035 -0.0009039130
## [13,]    13  24.0033 0.9995301 -0.00046988  -21.72562 -0.0008960584
## [14,]    14  27.6350 0.9995323 -0.00046768  -24.79018 -0.0008880882
## [15,]    15  31.4905 0.9995346 -0.00046542  -27.99159 -0.0008800000
## [16,]    16  35.5638 0.9995369 -0.00046311  -31.31741 -0.0008717910
## [17,]    17  39.8486 0.9995392 -0.00046075  -34.75519 -0.0008634586
## [18,]    18  44.3387 0.9995417 -0.00045833  -38.29250 -0.0008550000
## [19,]    19  49.0278 0.9995441 -0.00045585  -41.91689 -0.0008464122
## [20,]    20  53.9097 0.9995467 -0.00045330  -45.61593 -0.0008376923
## [21,]    21  58.9783 0.9995493 -0.00045069  -49.37717 -0.0008288372
## [22,]    22  64.2272 0.9995520 -0.00044802  -53.18817 -0.0008198438
## [23,]    23  69.6503 0.9995547 -0.00044527  -57.03649 -0.0008107087
## [24,]    24  75.2414 0.9995575 -0.00044246  -60.90970 -0.0008014286
## [25,]    25  80.9942 0.9995604 -0.00043956  -64.79535 -0.0007920000
## [26,]    26  86.9025 0.9995634 -0.00043659  -68.68100 -0.0007824194
## [27,]    27  92.9601 0.9995665 -0.00043354  -72.55420 -0.0007726829
## [28,]    28  99.1607 0.9995696 -0.00043041  -76.40253 -0.0007627869
## [29,]    29 105.4982 0.9995728 -0.00042718  -80.21353 -0.0007527273
## [30,]    30 111.9664 0.9995761 -0.00042387  -83.97477 -0.0007425000
## [31,]    31 118.5589 0.9995795 -0.00042045  -87.67381 -0.0007321008
## [32,]    32 125.2696 0.9995831 -0.00041694  -91.29820 -0.0007215254
## [33,]    33 132.0923 0.9995867 -0.00041332  -94.83551 -0.0007107692
## [34,]    34 139.0208 0.9995904 -0.00040959  -98.27329 -0.0006998276
## [35,]    35 146.0487 0.9995943 -0.00040575 -101.59911 -0.0006886957
## [36,]    36 153.1700 0.9995982 -0.00040179 -104.80051 -0.0006773684
## [37,]    37 160.3783 0.9996023 -0.00039770 -107.86508 -0.0006658407
## [38,]    38 167.6676 0.9996065 -0.00039348 -110.78035 -0.0006541071
```

```
## [39,]    39 175.0314 0.9996109 -0.00038912 -113.53389 -0.0006421622
## [40,]    40 182.4637 0.9996154 -0.00038462 -116.11326 -0.0006300000
## [41,]    41 189.9582 0.9996200 -0.00037996 -118.50603 -0.0006176147
## [42,]    42 197.5087 0.9996249 -0.00037515 -120.69974 -0.0006050000
## [43,]    43 205.1089 0.9996298 -0.00037016 -122.68196 -0.0005921495
## [44,]    44 212.7527 0.9996350 -0.00036500 -124.44024 -0.0005790566
## [45,]    45 220.4338 0.9996404 -0.00035964 -125.96216 -0.0005657143
## [46,]    46 228.1460 0.9996459 -0.00035409 -127.23526 -0.0005521154
## [47,]    47 235.8831 0.9996517 -0.00034833 -128.24710 -0.0005382524
## [48,]    48 243.6388 0.9996576 -0.00034235 -128.98525 -0.0005241176
## [49,]    49 251.4070 0.9996639 -0.00033614 -129.43726 -0.0005097030
## [50,]    50 259.1814 0.9996703 -0.00032967 -129.59070 -0.0004950000
```

In the table:

- *depth* is the depth of liquid above the bottom of the hemisphere and the measuring point for pressure
- deltaV is the liquid thermal expansion for 1 deg C
- Pr is the pressure ratio
- rP is the relative pressure ratio
- Ve is the total volumetric error indicated by the mass measurement error
- rVe is the relative volumetric error in the hemisphere

Plots

Figure 6 shows the increasing thermal expansion with depth. Note that Figure 7 shows the volumetric error reported by the mass measurement system increases in magnitude until it reaches the constant value of -129.6 cubic feet where it remains constant above the 50 foot depth (in the cylindrical portion). The relative volumetric error is shown in Figure 8. Finally, in Figure 9 we see the relative volumetric error in the hemispherical portion as well as the cyclindrical portion up to a depth of 200 feet. As previously indicated this relative error tends to approach zero as the ratio of the cylindrical portion to hemispherical depth increases.

Summary and Lessons Learned

We have seen how the behavior of liquids stored in a container with a hemispherical bottom behaves when subjected to temperature changes and specifically how the pressure measured at the bottom of the container can give distortion errors when using the mass measurement method. This is particularly important in the field of leak detection where a non leaking tank has a liquid content mass that is constant with time where the leak test requires isolation of the tank by making sure that all liquid streams into and out of the tank are entirely closed off.

In cylindrical tanks with essentially flat bottoms the error in the measurement of mass caused by thermal changes in the liquid is virtually non existent. Said another way, the thermal noise in the measurement cancels out because the thermal expansion volume ratio is exactly the reciprocal of the density ratio at different fluid temperatures over reasonably small temperature changes.

In a container without a flat bottom such as the hemispherical bottom we have examined we have learned that:

- Thermal gradients in containers without flat bottoms can have a substantial impact on the overall mass method method for leak detection testing methods.

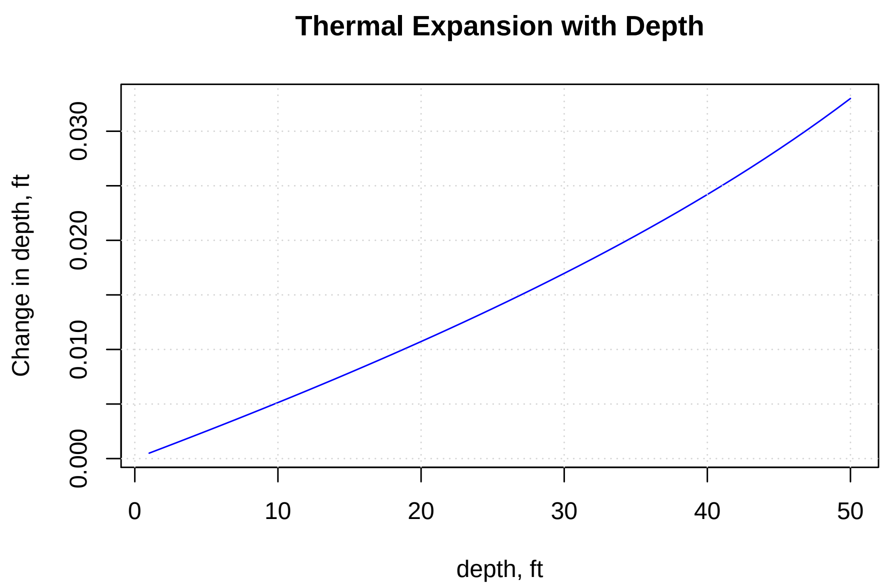

Figure 6: Variable Depth in the Hemispherical Portion

- The thermally induced error occurs only in the non cylindrical section at the bottom.
- A temperature increase in the tank bottom shows a mass loss or volumetric loss in the tank; a temperature decrease would show the opposite.
- For any container where the bottom radius is less than the cylindrical radius the effects will be similar to those presented. Such bottoms may be conical, spherical caps, or other shapes.
- If the highest possible leak detection probabilities with the smallest false positive rates are desired using mass leak detection methods, then protocols which change the temperature in these tanks is required for leak detection test validation.
- In this paper we have assumed that thermal changes are uniform. In real containers the gradients may be many different patterns. The magnitude and characteristic effects of noise generated by these gradients is not known. Because thermal gradients can move can cause flow within the container, we have assumed static conditions and have not taken into account any dynamic effects which are expected to be small compared to those effects discussed in this paper.

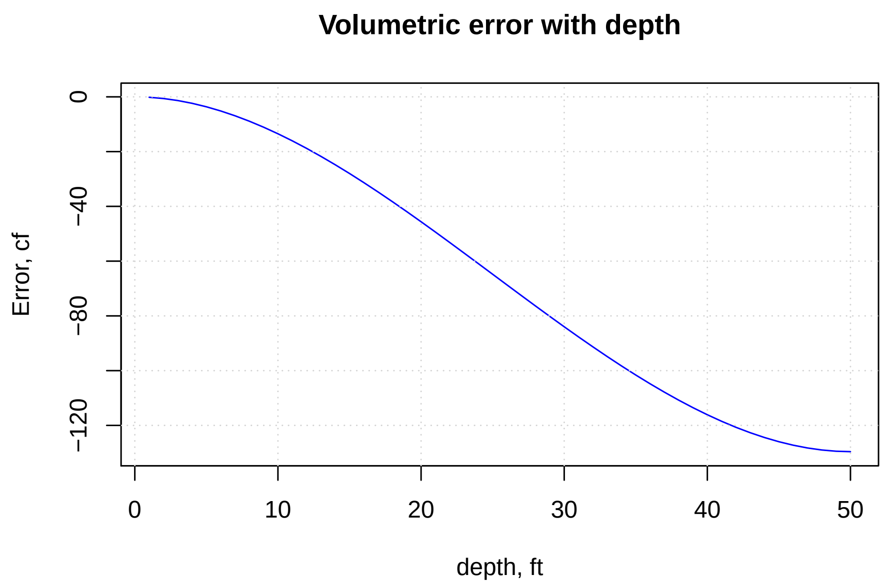

Figure 7: Variable Depth in the Hemispherical Portion

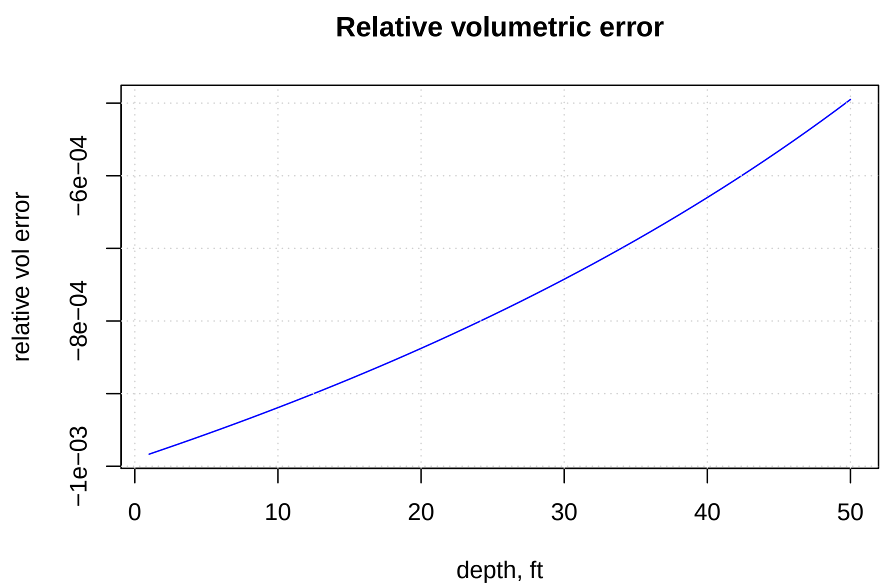

Figure 8: Variable Depth in the Hemispherical Portion

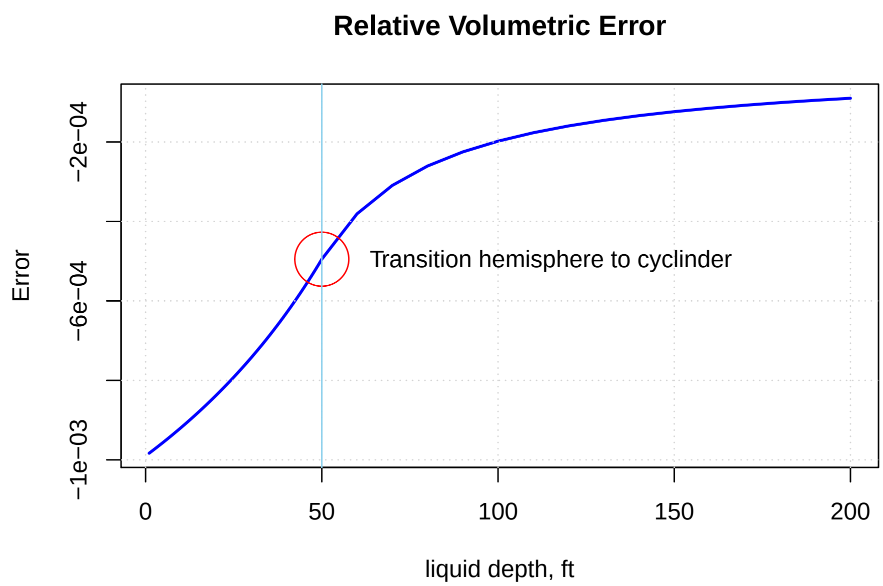

Figure 9: Variable Depth in the Hemispherical Portion