Toronto Math Circles: Junior
Fourth Annual Christmas Mathematics Competition
Saturday, December 16, 2017
1:00 pm - 3:00 pm

Each problem is graded on a basis of 0 to 10 points. All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Some partial credit may be given, but only when a contestant has shown significant and substantial progress toward a solution. Calculators are not allowed.

1. A four digit number A, when read from left to right, consists of 4 consecutive increasing integers. Determine the sum of the possible values of A.

2. Determine if there exists a 3 digit number n, such that $600 \leq n \leq 699$ and removing the ten's digit yields a two digit number that is 6 less than $\frac{1}{6}$ of the original number.

3. Two rectangles are drawn on the plane such that one rectangle is entirely contained in the other. Describe a straight edge construction of a line that equally divides the area of the region between the two rectangles. Be sure to explain why your construction works.

4. A sequence of number is called **COOL** if every term starting from the third term is the average of all the previous terms. A particular COOL sequence has the first term as 1 and the 2017^{th} term as 2017. Determine the value of the second term.

5. On the Cartesian plane, each lattice point is assigned an integer such that, for any square with its sides parallel to two axis, the sum of values at the vertices is equal to the number of lattice points in this square. For example, $(-1,1),(1,1),(1,-1),(-1,-1)$ has 9 lattice points inside the square. Prove that no such assignment exists. Note: A lattice point is a point (x,y) such that x and y are both integers.

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1. Let a and b be two numbers such that $b = a+2$. Consider two monic quadratics such that their vertices are at $A(a, 0)$ and $B(b, 0)$. Given that these two quadratics only intersect at one point, C. Determine the area of the triangle ABC.

2. On the Cartesian plane, each lattice point is assigned an integer value such that, for any square with positive area and its four vertices on lattice points, the sum of values at the vertices is equal to the area of the square. Prove that no such assignment exists. Note: A lattice point is a point (x, y) such that x and y are both integers.

3. Let n be an odd positive integer not divisible by 5. Prove that in the following n numbers

$$1, 11, \ldots, \underbrace{11\cdots11}_{n}$$

there is one that is divisible by n.

4. Let AB be the diameter of a semi-circle with center O. Point C is the midpoint of arc AB and M is the midpoint of the chord AC. If CH is perpendicular to BM at H, prove that $CH^2 = AH \cdot OH$. Be sure to include a clearly labeled diagram.

5. Let c and d be integers with $c \neq 0$. Prove that c divides d if and only if there are infinitely many distinct integral pairs (x, y) such that x and y both divide $c(x + y) + d$.