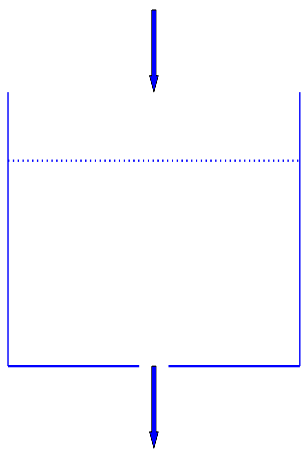
Math 3331 - Spring 2024 - Mixtures

Example. 1. A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Water that contains 1/2 pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate. Find the amount of salt at any time t.



$$\frac{dA}{dt} = A_{in} - A_{out}$$
$$= r_i c_i - r_o c_o$$

$r_i = 4\,\text{gal/min}$ $\qquad$ $c_i = 1/2\,\text{lb/gal}$

$r_o = 4\,\text{gal/min}$ $\qquad$ $c_o = \dfrac{A}{600}\,\text{lb/gal}$

So ODE is

$$\frac{dA}{dt} = 4\,\frac{gal}{min} \cdot \frac{1}{2}\,\frac{lb}{gal} - 4\,\frac{gal}{min} \cdot \frac{A(t)}{600}$$
$$= 2 - \frac{A}{150}$$

with the initial condition $A(0) = 40$. Now we solve

$$\frac{dA}{dt} + \frac{A}{150} = 2 \quad \text{(linear)}$$

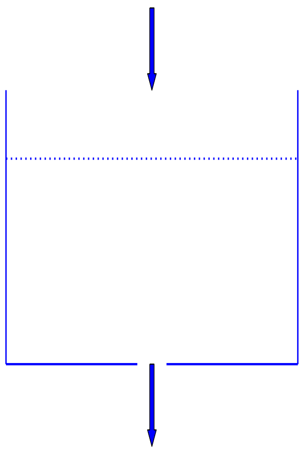
Integrating factor $\mu = exp(\int \frac{1}{150}\,dt) = exp\left(\frac{t}{150}\right)$. So

$$\frac{d}{dt}\left(exp\left(\frac{t}{150}\right)A\right) = 2exp\left(\frac{t}{150}\right) \quad \to \quad exp\left(\frac{t}{150}\right)A = 300exp\left(\frac{t}{150}\right) + c$$

Now $A(0) = 40$ gives $c = 40 - 300 = -260$ and we have

$$A = 300 - 260\,exp\left(\frac{-t}{150}\right)$$

Example. 2. A 500 liter tank initially contains 10g of salt dissolved in 200 liters of water. Water that contains 1/4 g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min. Find the salt at any time.



$$\frac{dA}{dt} = A_{in} - A_{out}$$
$$= r_i c_i - r_o c_o$$

Note: This tank is filling up so the volume is changing. Here

$$V(t) = (r_i - r_o)t + C_0 = (4-2)t + 200 = 2t + 200$$

$r_i = 4\,\text{gal/min}$ $\quad\quad c_i = 1/4\,\text{lb/gal}$

$r_o = 2\,\text{gal/min}$ $\quad\quad c_o = \dfrac{A}{2t+200}\,\text{lb/gal}$

So ODE is

$$\frac{dA}{dt} = 4\,\frac{gal}{min}\cdot\frac{1}{4}\,\frac{lb}{gal} - 2\,\frac{gal}{min}\cdot\frac{A(t)}{2t+200}$$
$$= 1 - \frac{A}{t+100}$$

with the initial condition $A(0) = 10$. Now we solve

$$\frac{dA}{dt} + \frac{A}{t+100} = 1 \quad \text{(linear)}$$

Integrating factor $\mu = exp(\int \frac{1}{t+100}\,dt) = exp\,(\ln(t+100) = t+100$. So

$$\frac{d}{dt}\left((t+100)A\right) = t + 100 \quad \rightarrow \quad (t+100)A = \frac{1}{2}(t+100)^2 + c$$

Now $A(0) = 10$ gives $c = -4000$ and we have

$$A = \frac{t+100}{2} - \frac{4000}{t+100}$$