Spring 2026 − Math 3331− M of UD

The solution of linear nonhomgeneous ODEs

$$a(x)y'' + b(x)y' + c(x)y = f(x) \tag{1}$$

comprises of two parts

$$y = y_h + y_p \tag{2}$$

The y_h is the solution of the homogeneous problem

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{3}$$

and y_p some solution of the entire problem (called the particular solution).

1. Underdetermined Coefficients

We first consider

$$y'' - 3y' + 2y = 2x + 1 \tag{4}$$

The homogeneous solution is

$$y_h = c_1 e^x + c_2 e^{2x} \tag{5}$$

Now for the particular solution. If we guessed

$$y = 2x + 1 \tag{6}$$

this won't work as

$$-4 \neq 2x + 1 \tag{7}$$

So without actually guess we look for a solution of the form

$$y_p = Ax + B \tag{8}$$

where A and B are to be determined. Substituting into the ODE gives

$$0 - 3(A) + 2(Ax + B) = 2x + 1 \tag{9}$$

and we compare terms, the x terms and the q terms. This gives

$$2A = 2 \quad -3A + 2B = 1 \quad \rightarrow \quad A = 1,\ B = 2 \tag{10}$$

so

$$y_p = x + 2 \tag{11}$$

and the general solution is

$$y = c_1 e^x + c_2 e^{2x} + x + 2 \tag{12}$$

Example 2.

We first consider

$$y'' - 2y' + y = \sin x \tag{13}$$

The homogeneous solution is

$$y_h = c_1 e^x + c_2 x e^x \tag{14}$$

Now for the particular solution, if we try to guess

$$y = A \sin x \tag{15}$$

we have a problem. Substituting into (36) gives

$$-A \sin x - 2A \cos x + A \sin x = \sin x \tag{16}$$

There is simply no A that will work. However, if we use

$$y = A \sin x + B \cos x \tag{17}$$

then (36) becomes

$$-A \sin x - B \cos x - 2\left(A \cos x - B \sin x\right) + A \sin x + B \cos x = \sin x \tag{18}$$

and we compare terms, the $\sin x$ and $\cos x$ terms. This gives

$$-A + 2B + A = 1 \quad -B - 2A + B = 0 \quad A = 0, \quad B = 1/2 \tag{19}$$

so

$$y_p = \tfrac{1}{2} \cos x \tag{20}$$

and the general solution is

$$y = c_1 e^x + c_2 x e^x + \tfrac{1}{2} \cos x \tag{21}$$

So are there any limitations with this method? Well, it comes down to "good guessing". For example, consider

$$y'' - y' = 2? \tag{22}$$

The CE is

$$m^2 - m = 0 \tag{23}$$

so $m = 0$ and $m = 1$ and the complimentary solution is

$$y_c = c_1 + c_2 e^x \tag{24}$$

Since the RHS of (36) is 2, a natural guess is

$$y_p = A \tag{25}$$

however, substitution into (36) gives

$$0 = 2? \tag{26}$$

so certainly there is a problem. Interesting enough this ODE is independent of y and we can use reduction of order to solve it. So if $y' = u$ then $y'' = u'$ and (36) becomes

$$u' - u = 2? \quad \text{(linear and separable)} \tag{27}$$

and can be solved all the way through giving

$$y = -2x + c_1 + c_2 e^x \tag{28}$$

and one will notice the y_p we're looking for is

$$y_p = -2x. \tag{29}$$

If we consider

$$y'' - y' = e^x? \tag{30}$$

again RHS of (36) we look for particular solution of the form

$$y_p = Ae^x \tag{31}$$

gives

$$0 = e^x? \tag{32}$$

however, reduction of order gives

$$y = 2xe^x + c_1 + c_2 e^x \tag{33}$$

and $y_p = 2xe^x$. So a recap. We look at the right hand side (RHS) of the nonhomogeneous equation and guess. If the RHS has terms that belong to the complimentary solution, it necessary to bump up the solution with $x's$.

Ex 4 Find the general solution to

$$y'' - 2y' + y = 2e^x? \tag{34}$$

The CE is

$$m^2 - 2m + 1 = 0 \tag{35}$$

so $m = 1$ and $m = 1$ and the complimentary solutions is

$$y_c = c_1 x e^x + c_2 e^x \tag{36}$$

The RHS of (46) is $2e^x$. As the complimentary solution has both e^x and xe^x it necessary to bump twice and thus we look for a solution like

$$y_p = Ax^2 e^x \tag{37}$$

Substitution into (48) gives

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = 2e^x \tag{38}$$

and after cancellation

$$2Ae^x = 2e^x \tag{39}$$

so $A = 1$ and the general solution is

$$y = c_1 x e^x + c_2 e^x + x^2 e^x. \tag{40}$$

So are there limitations to this method. Well yes! It involves good guessing. So what about something like

$$y'' + y = \tan x \tag{41}$$

or

$$y'' + y = \frac{1}{x}? \tag{42}$$