Math 6345 – Adv. ODE's

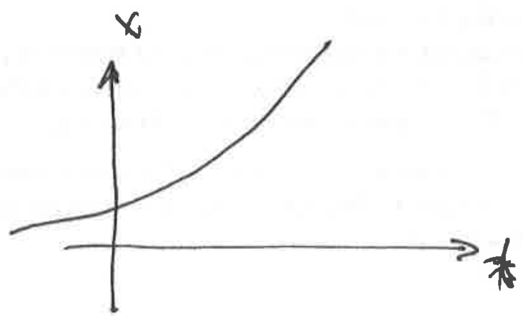
Stability Theory

So in ODEs 1 and in much of the course so far, we have been solving ODE / systems. Now we want characterize their sol^n's

Ex $\frac{dx}{dt} = x \qquad x(0) = 1$

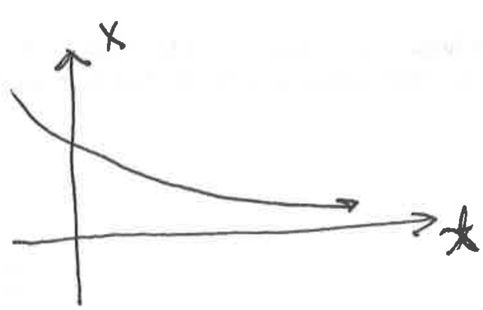
$$\frac{dx}{x} = dt \Rightarrow \ln x = t + \ln c \Rightarrow x = ce^{t}$$

$$x(0) = 1 \Rightarrow c = 1 \text{ so } \quad x = e^{t}$$



$$\lim_{t\to\infty} x = \lim_{t\to\infty} e^{t} \to \infty$$

Similar $\frac{dx}{dt} = -x \qquad x(0) = 1 \qquad$ gives $\quad x = e^{-t}$



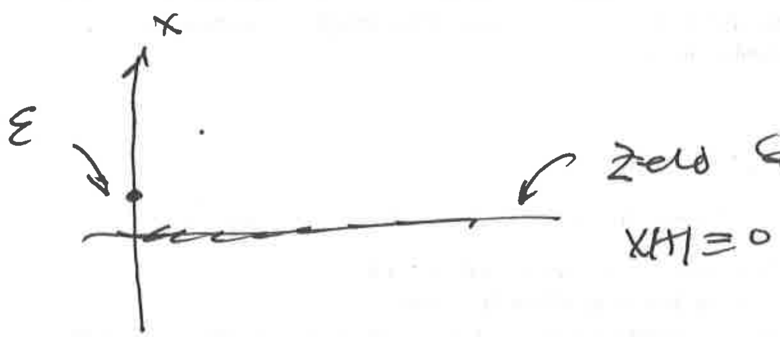
$$\lim_{t\to\infty} x = \lim_{t\to\infty} e^{-t} = 0$$

Note: For each $x(t) \equiv 0$ identically

and for the second $\lim_{x \to 0} x(t) \to 0$ the zero sol^n

if this happens we say zero sol^n is "asymptotically" stable.

what it really means if we perturb a small about from the zero sol^n



then $x(t) = \varepsilon e^{-t}$

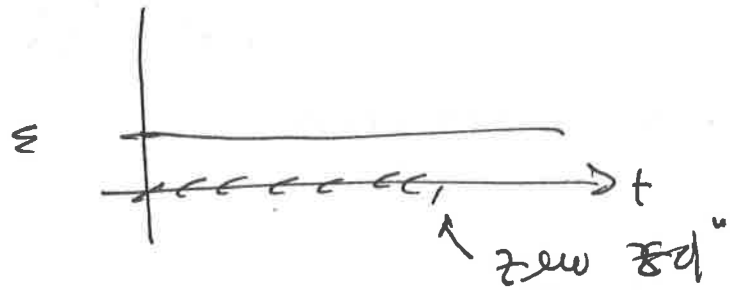
and $\lim_{t \to \infty} x(t) \to 0$ we return back to the zero sol^n

How about

$\dot{x} = 0$. The zero sol^n is still a sol^n

and a small perturbation ε

$x = \varepsilon$



and we see we remain at most ε from the zero sol^n. So when we start near the zero sol^n and remain near the zero sol^n, the zero sol^n is said to be "stable". (Not asymptotically)

The first sol^n is just "unstable".

<u>Linear Systems</u>

We now consider

$$\frac{d\bar{x}}{dt} = A\bar{x}$$ where A is a 2×2 constant coefficient matrix

so let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

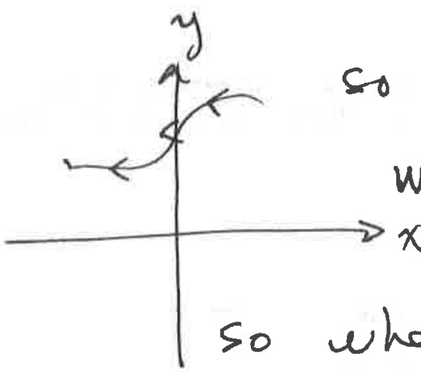
the eigenvalues are determined by $|\lambda I - A| = 0$

or $\lambda^2 - (a+d)\lambda + ad - bc = 0$

and can either be (i) real & distinct (ii) real repeated or (iii) complex

we want to know the stability of the zero sol^n $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

To do this we consider the phase plane the sol^n $x = x(t)$, $y = y(t)$ plotted x vs y



so as t increases (or decrease) we move along this trajectory

so what happens when we leave the zero sol^n $(0,0)$. Will we return, stay close by or never get close again. The eigenvalues will determine this.

We consider the various possibilities by example

(i) Real Distinct Eigenvalues
- both negative

Ex!

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\bar{x}$$

$$\begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda+2 \end{vmatrix} = 0 \qquad (\lambda+1)(\lambda+2) = \qquad \lambda = -1, -2$$

$$\lambda = -1 \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} u \text{ any thing} \\ v = 0 \end{matrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = -2 \qquad \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} u = 0 \\ v \text{ anything} \end{matrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

sol^n $$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-2t}$$

or $$x = c_1 e^{-t}, \quad y = c_2 e^{-2t}$$

If we eliminate $\quad x^2 = c_1^2 e^{-2t}, \quad y = c_2 e^{-2t}$

$$\frac{y}{x^2} = \frac{c_2}{c_1^2} \quad \text{or} \quad y = kx^2$$ trajectories in (x, y) plane

1st pull out 2 special trajectories

(1) start at $x=1, y=0$ at $t=0$

then $x = c_1 e^{-t} \Rightarrow 1 = c_1$

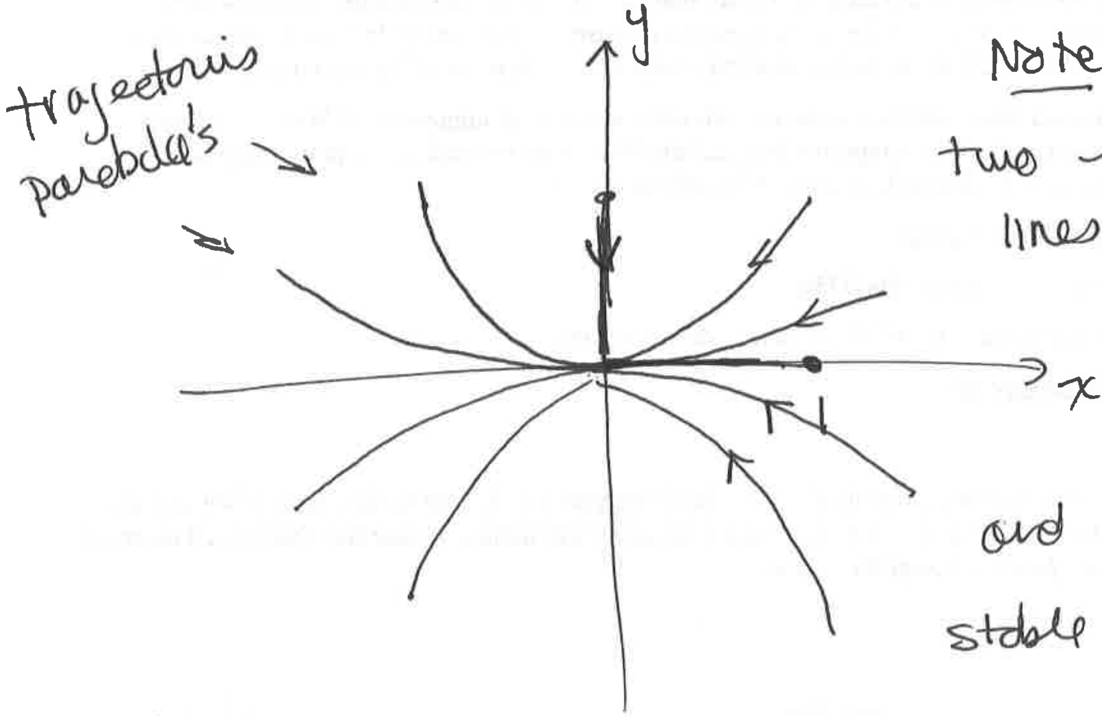
$y = c_2 e^{-2t} \Rightarrow 0 = c_2$

so $x = e^{-t}, \; y = 0$

(2) start at $x=0, y=1$ at $t=0$

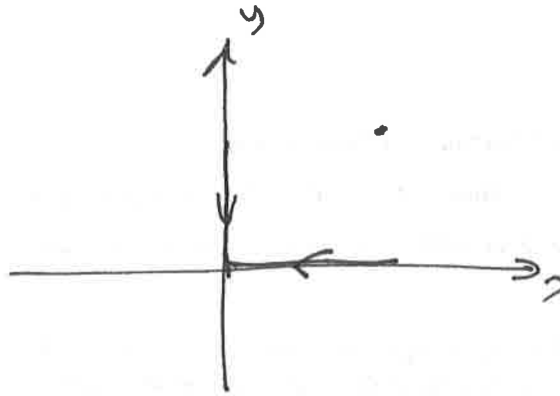
then $c_1 = 0, \; c_2 = 1$

and $x = 0, \; y = e^{-2t}$



Note: These two special lines coincide with the eigenvector $\begin{pmatrix}1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\1\end{pmatrix}$ and are called stable manifolds.

so if we are at a point say (1,1)



which of the two manifolds are we attracted to the most

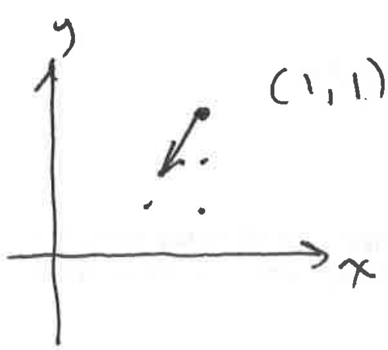
Recall $\lambda = -1$ $\bar{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\lambda = -2$ $\bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

sol^n $\bar{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$

as $t \to \infty$ or increases, the second term dies off faster leaving $\bar{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$

so it is this one we approach.

Also if $\dot{x} = -x$, $\dot{y} = -2y$ that at (1,1)



(1,1) $\dot{x} = -1$, $\dot{y} = -2$

and y is decreasing at a faster rate.

Ex 2 $\quad \frac{d\bar{x}}{dt} = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix} \bar{x}$

Eigenvalues $\quad \begin{vmatrix} \lambda+3 & -1 \\ -2 & \lambda+2 \end{vmatrix} = 0 \qquad (\lambda+2)(\lambda+3) - 2 = 0$

$$\lambda^2 + 5\lambda + 4 = 0$$

$$(\lambda+1)(\lambda+4) = 0$$

so $\lambda = -1, -4$

Eigenvectors

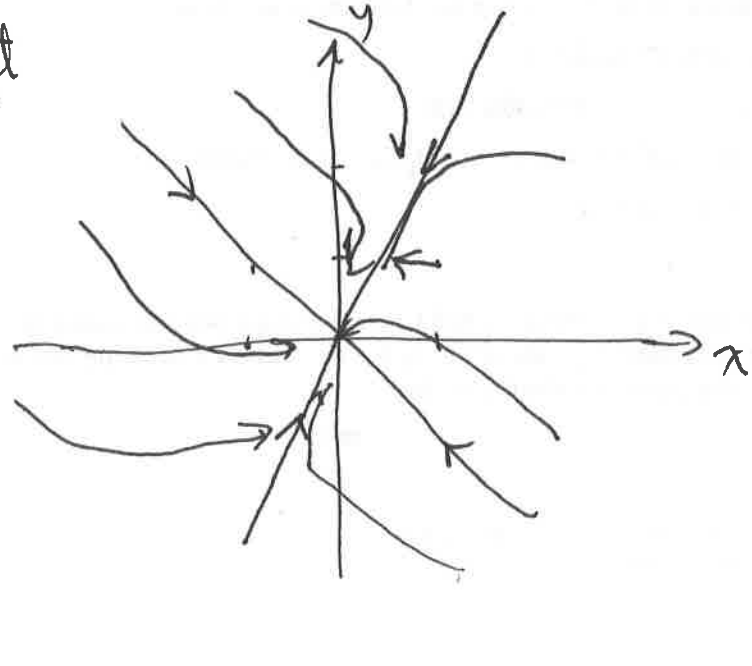
$\lambda = -1 \qquad \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 2u - v = 0$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\lambda = -4 \qquad \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad u + v = 0$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Phase Portrait



Pick point $(1,1)$

$$\frac{dx}{dt} = -3(1) + 1 = -2$$

$$\frac{dy}{dt} = 2(1) - 2(1) = 0$$

Note if we let $T = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

then $\det T = -3$

$$T^{-1} = -\frac{1}{3}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

and

$$T^{-1} D T = A \quad \text{where} \quad D = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$$

so if $\dot{\bar{x}} = A\bar{x}$ then $\dot{\bar{x}} = T^{-1} D T x$

$$\Rightarrow \quad T\dot{\bar{x}} = DT\bar{x}$$

and under a change of variables

$$\bar{y} = T\bar{x} \quad \text{gives} \quad \dot{\bar{y}} = D\bar{y}$$

so $u = x + y$
$v = 2x - y$

and the two stable manifolds $y = -x$ & $y = 2x$ become $u = 0$ & $v = 0$