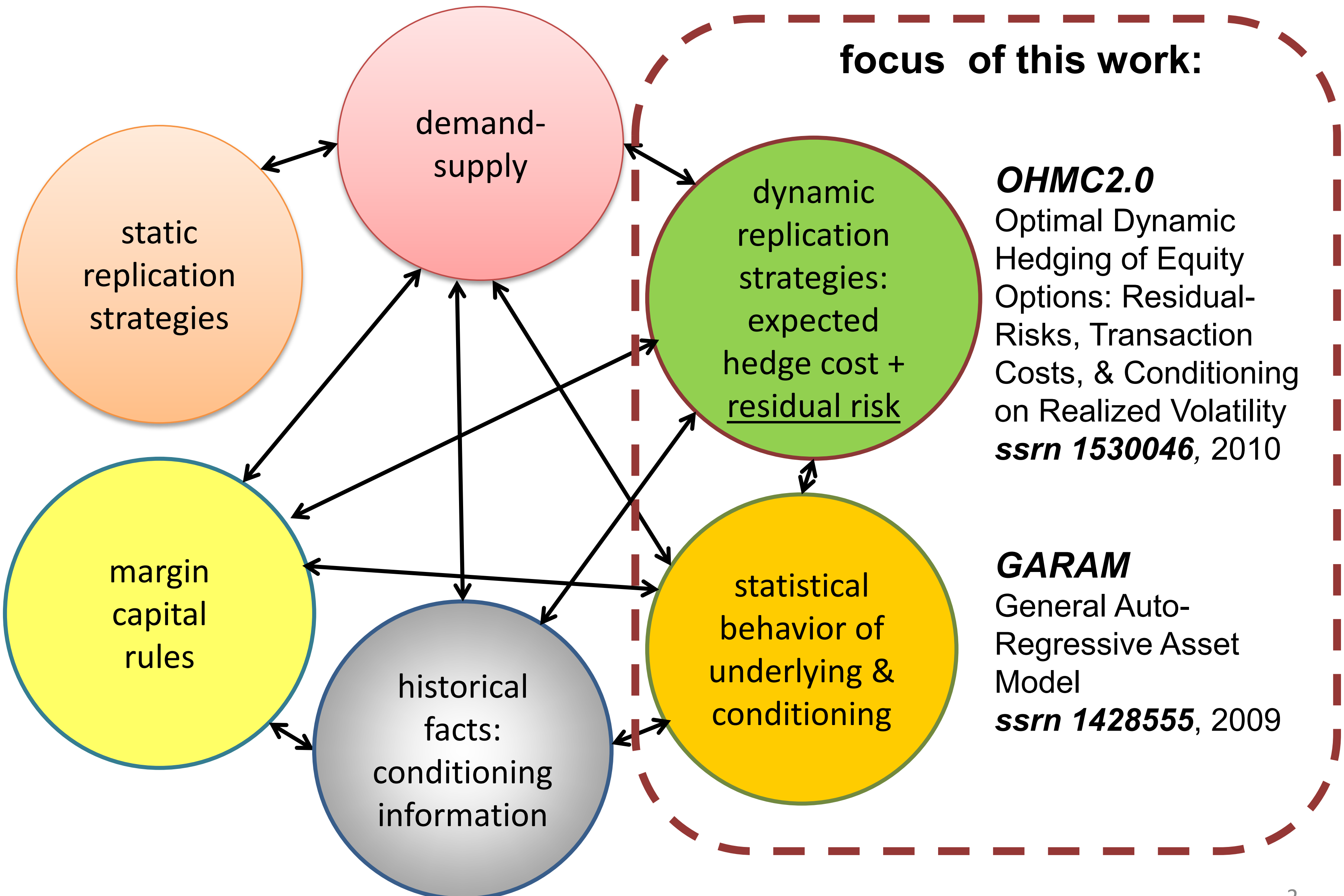
Optimal Dynamic Hedging of Equity Options
Residual Risks, Transaction Costs & Conditioning

Vivek Kapoor

vivek2.kapoor@citi.com
Citi

Global Derivatives Trading & Risk Management
May 2010

Real Option Price Dynamics



A Trader's View of Option Prices

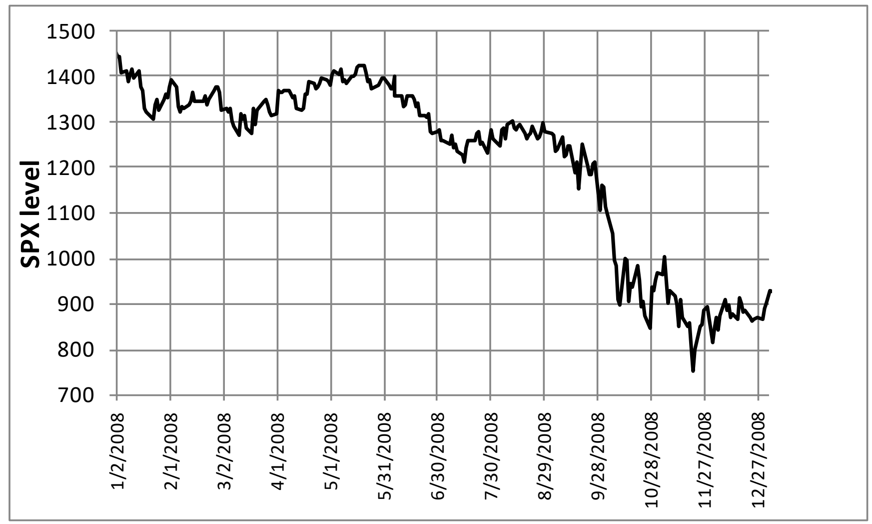
Option Sell Price	=	Expected Hedging Cost *expected negative P&L incurred by dynamic hedging*	+	Residual Risk Premium *expected net P&L of option seller-dynamic hedger*

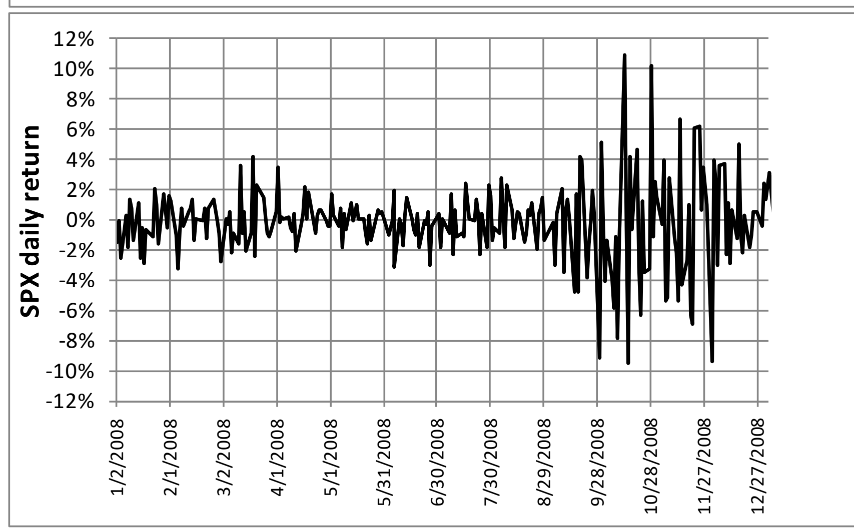
- Residual risks dictate option seller hedger 's capital requirements
- Provider of capital-margin have expectations of return!
- **Option prices reflect expected hedging costs & premiums for residual risks**
- An informed view of the option trading risk-premium can be used to develop trading strategies that are aided by
  - **realistic statistical description of underlying**
  - **explicit delineation of hedging strategy**

Statistical Behavior of Underlying

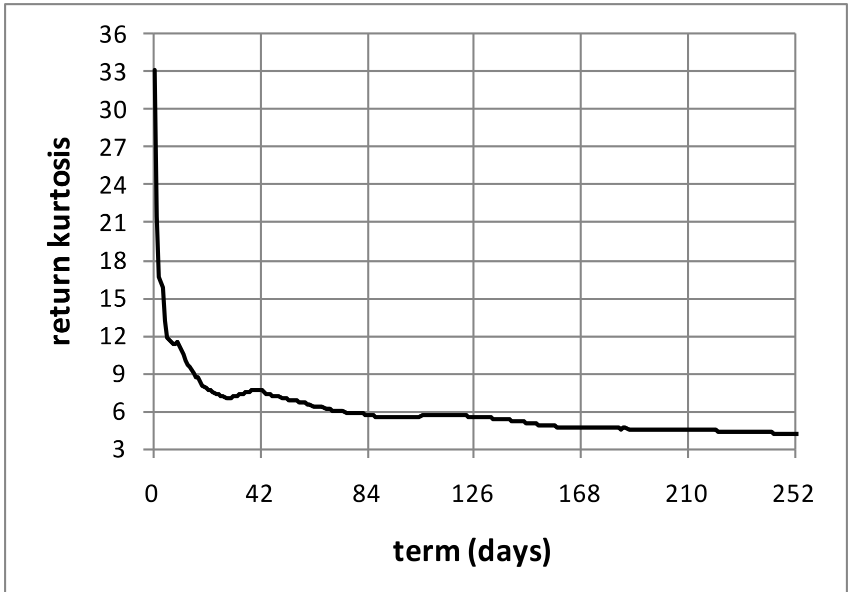
Empirical features of SPX are employed to illustrate the applicability of GARAM & OHMC2.0 to SPX option in this work.

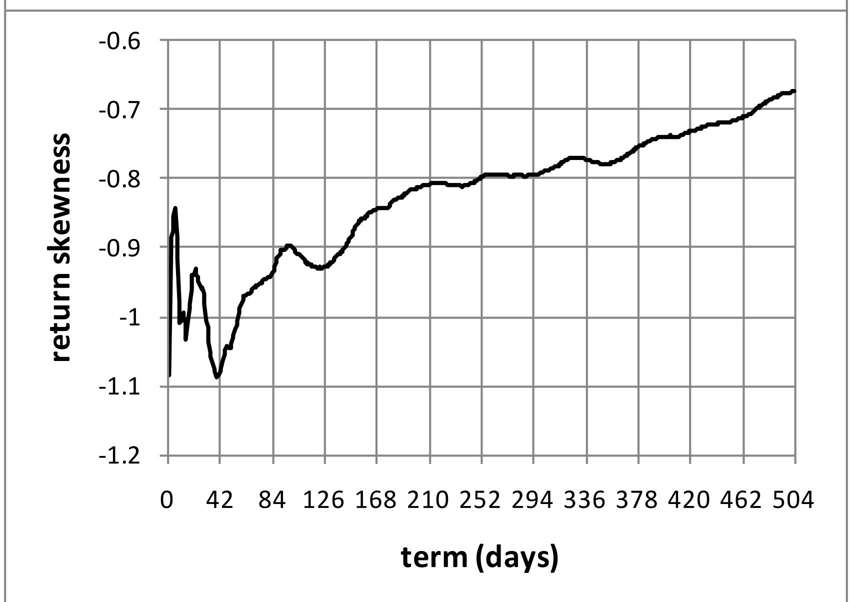
- **Heteroskedasticity**
- **Fat tails**
- **Asymmetry**
- **Multiple time-scales**

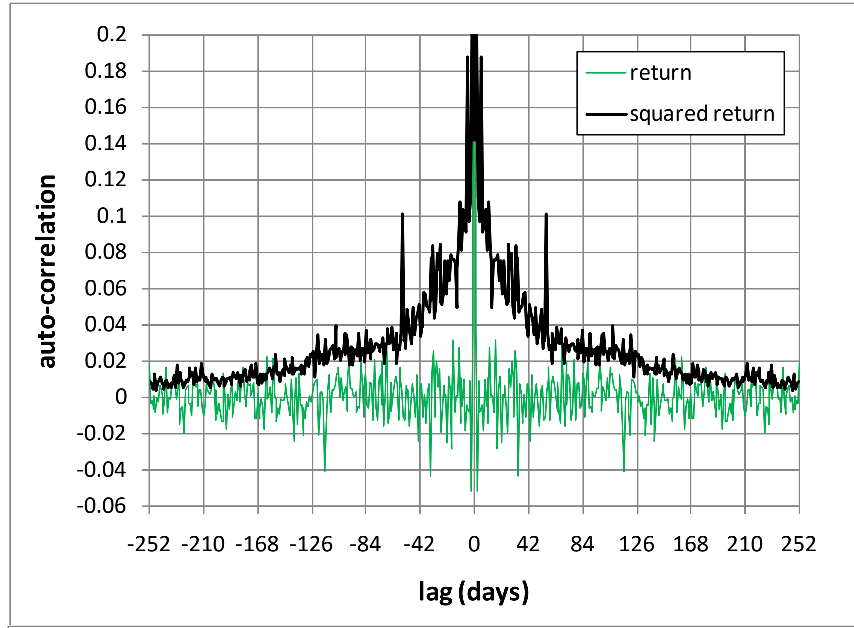


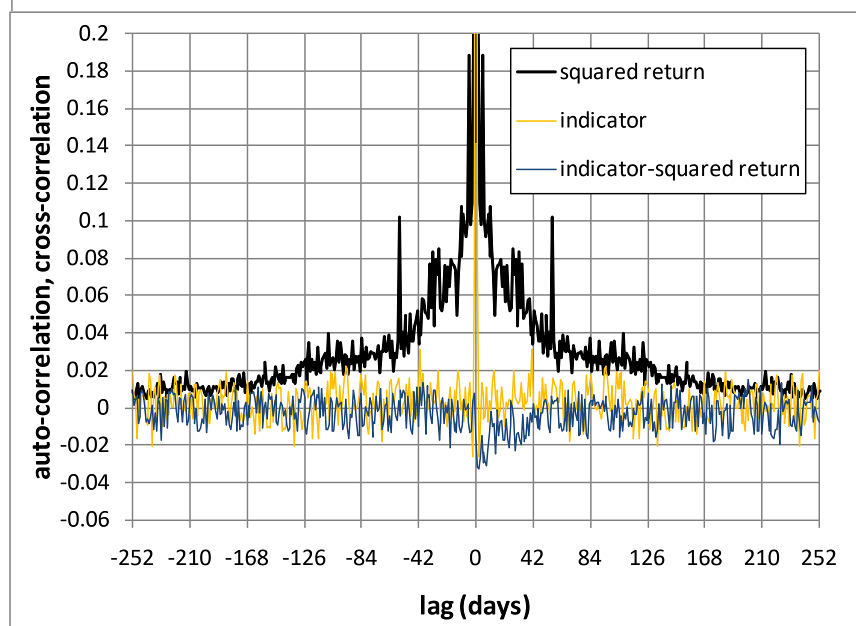


Empirical Features of Equity Index Returns









General Auto-Regressive Asset Model

GARAM Specification

$$x = \ln\left(r^2 / R^2\right)$$

$$y = x\left[1 + p_1\left(\pi/2 + Tan^{-1}\left[p_2\left(x + p_3\right)\right]\right)\right];\ p_1 \geq 0$$

$$\left(r/R\right)^2 = F\left(y\right)$$

$$\rho_{yy}\left(\tau\right) \equiv E\left[\left(y(t) - \bar{y}\right)\left(y(t+\tau) - \bar{y}\right)\right] / \sigma_y^2$$

$$I(t) = \begin{cases} +1 & \text{if } z < z_I \\ -1 & \text{if } z \geq z_I \end{cases}$$

$$\rho_{zz}\left(\tau\right) \equiv E\left[\left(z(t) - \bar{z}\right)\left(z(t+\tau) - \bar{z}\right)\right] / \sigma_z^2$$

$$\rho_{zy}\left(\tau\right) \equiv E\left[\left(z(t) - \bar{z}\right)\left(y(t+\tau) - \bar{y}\right)\right] / \left(\sigma_z \sigma_y\right)$$

$$r(t) = I\left(z(t)\right) R \left|\sqrt{F\left(y(t)\right)}\right|$$

- GARAM models the return **magnitude** & **sign** as functions of two stationary correlated stochastic processes

- The term-structure of non-normality and the temporal scales of fluctuations are captured by *empirical* auto & cross covariance functions

- Well developed **simulation** & **filtering** methodologies can be applied to create a realistic **conditional description** of the underlying using GARAM

http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1428555

General Auto-Regressive Asset Model

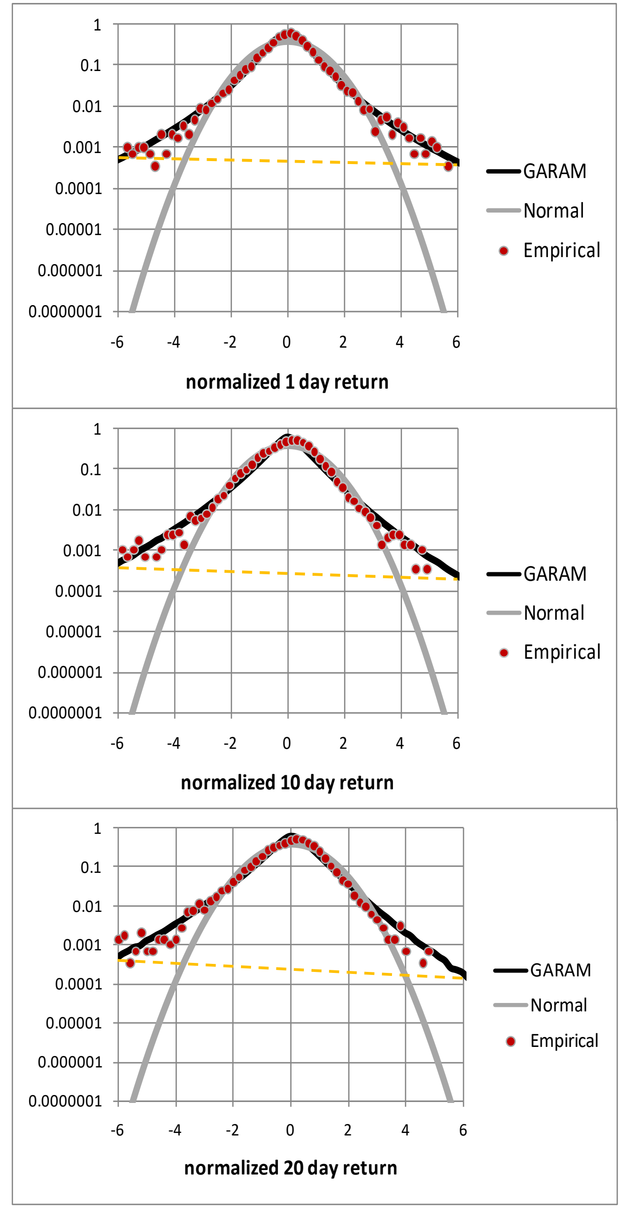
Probability Density of SPX Returns: Empirical, GARAM, & Normal Distribution

The Normal distribution is clearly unacceptable! The build-up of asymmetry in GARAM is accentuated by the dashed yellow line, that would be horizontal for a symmetric return distribution.

This build-up of asymmetry is driven by the covariance between return sign indicator and future return magnitude, in the GARAM model. This has been called the *leverage-effect*.

The term-dependence of return kurtosis in GARAM is driven by the auto-covariance of return magnitude (squared return).

(SPX returns: January 3, 1950 - June 2, 2009)



Optimal Hedge Monte-Carlo (OHMC) Method

Cost function

$$C\left(s(t_k),\varpi_1(t_k),\varpi_2(t_k),\ldots\ldots\ldots,\varpi_J(t_k),t_k\right)$$

spot value   conditioning variables

$$\phi\left(s(t_k),\varpi_1(t_k),\varpi_2(t_k),\ldots\ldots\ldots,\varpi_J(t_k),t_k\right)$$

Hedge ratio function

Option seller-hedger's wealth change:

$$\Delta W_{t_k}^{option}(t_k,t_{k+1}) = C\left(s(t_k),\varpi(t_k),t_k\right) - G(t_k)$$

$$G(t_k) = C\left(s(t_{k+1}),\varpi(t_{k+1}),t_{k+1}\right)df(t_k,t_{k+1}) + P(t_{k,i})df(t_k,t_{k,i})$$

$$\Delta W_{t_k}^{hedge}(t_k,t_{k+1}) = \phi\left(s(t_k),\varpi(t_k),t_k\right)H(t_k)$$

$$H(t_k) = \left[s(t_{k+1}) - \frac{s(t_k)}{DF(t_k,t_{k+1})}\right]df(t_k,t_{k+1}) + \pi_i df(t_k,t_i)$$

$$\Delta W_{t_k}^{tc}(t_k,t_{k+1}) =$$
$$-\left[\delta\left|\phi\left(s(t_{k+1}),\varpi(t_{k+1}),t_{k+1}\right) - \phi\left(s(t_k),\varpi(t_k),t_k\right)\right| + \chi\right]df(t_k,t_{k+1})$$

$$\Delta W_{t_k}(t_k,t_{k+1}) = \Delta W_{t_k}^{option}(t_k,t_{k+1}) + \Delta W_{t_k}^{hedge}(t_k,t_{k+1}) + \Delta W_{t_k}^{tc}(t_k,t_{k+1})$$

OHMC solves for the unknown functions $C\left(s(t_k),\varpi(t_k),t_k\right)$ $\phi\left(s(t_k),\varpi(t_k),t_k\right)$ so that:

$$E[\Delta W_{t_k}(t_k,t_{k+1})] = \overline{\Delta W_{t_k}(t_k,t_{k+1})}$$

$$\text{minimize} \quad \sigma^2_{\Delta W_{t_k}(t_k,t_{k+1})} \equiv E[(\Delta W_{t_k}(t_k,t_{k+1}) - \overline{\Delta W_{t_k}(t_k,t_{k+1})})^2]$$

http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1530046

OHMC SAMPLE ANALYSIS PARAMETERS

Tenor Range:	10, 21, 42 trading days				
Options & Strikes:	Puts	100%	95%	85%	
	Calls	100%	105%	115%	
Initial Volatility Regime:	Low, High				
Position:	Option Seller-Hedger, Option Buyer-Hedger				
Hedging Mode:	**(a)**	**(b)**	**(c)**	**(d)**	**(e)**
Transaction Costs	off	***on***	off	off	***on***
*Local Risk Premium**	off	off	***on***	off	***on***
*Hedge Conditioning***	off	off	off	***on***	***on***

*The mean change in wealth constraint of OHMC is used to incorporate a risk-premium *locally* in time. Specifically, the annualized Sortino ratio over the hedging interval (daily in our examples) is set to 1.

**I condition hedging and valuation on the 10-day trailing realized volatility, in addition to the spot value.

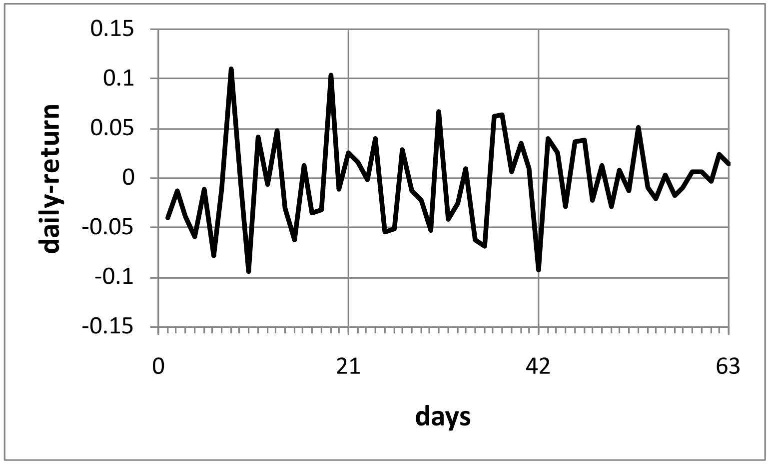
CONDITIONING INFORMATION

High-Volatility Conditioning Return Data

The return volatility over the 63 day conditioning window is 67.6% and over the last 10 days 21.6%.

Dividend payments are assumed to be constant and paid uniformly every day over the option horizon (10-42 days) and are taken as 3.28%/yr of spot value at inception.

Fixed transaction costs are set to zero and the per-share transaction cost (includes commissions and spread from *mid* price) = 2.77 bps of spot at inception.

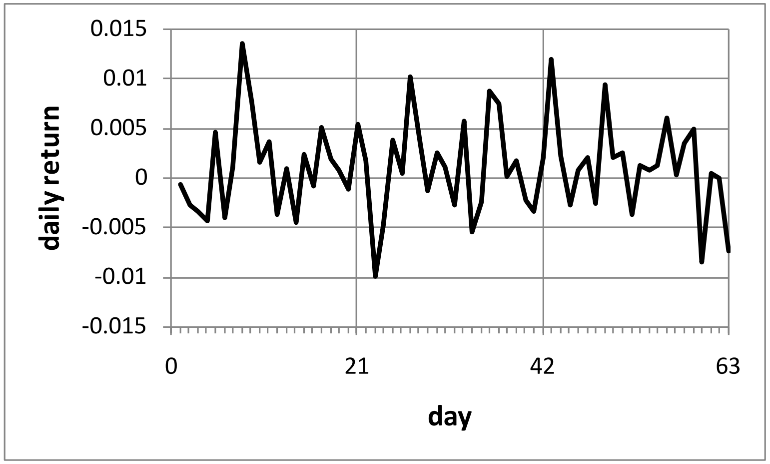


Low Volatility Conditioning Return Data

The return volatility over the 63 day conditioning window is 7.56% and over the last 10 days 7.15%.

Dividend payments are assumed to be constant and paid uniformly every day over the option horizon (10-42 days) and are taken as 1.78%/yr of spot value at inception.

Fixed transaction costs are set to zero and the per-share transaction cost (includes commissions and spread from *mid* price) = 1.83 bps of spot at inception.



OPTION PRICING DYNAMICS

Intuition about option trading can be built based on the framework advanced here by recognizing the roles of the following:

- realized volatility – long term & short term
- fluctuations and autocorrelation of realized volatility
- cross-correlation of return sign-indicator & realized volatility
- average hedge costs

- when the average hedge P&L = 0 then the value function is the average hedge cost

-when a risk-premium constraint is embedded then the value function is the average hedge costs plus the risk premium

- hedge slippage metrics and their strike and term dependence
- inferred option price implied risk premium

HEDGE PERFORMANCE

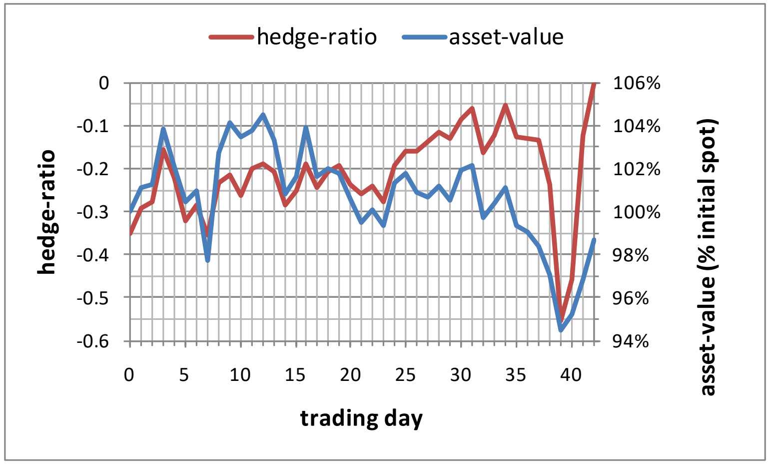
Median total P&L sample path hedge performance from OHMC analysis of sell-hedge 42 day 95% strike put in high volatility regime (hedge mode e)

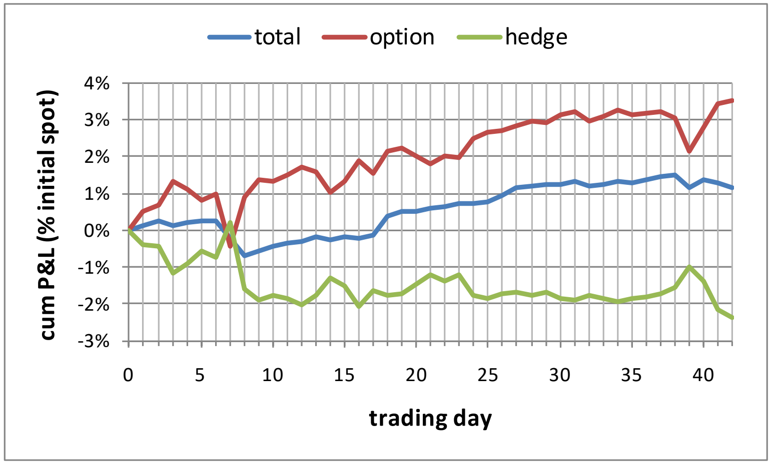
The hedging strategy is cognizant of transaction costs and is conditioned on the trailing 10 day realized volatility.

The hedging strategy also seeks to maintain an *expected* P&L over each hedging interval to achieve a target Sortino-Ratio of 1.

The attempted replication is full of slips even in this relatively benign outcome. The risk-premium charged by the seller hedger towards the goal of maintaining a Sortino-Ratio of 1 every day is fulfilled insofar as the total P&L at the end of 42 days slightly exceeds the initially expected P&L in pricing the put.

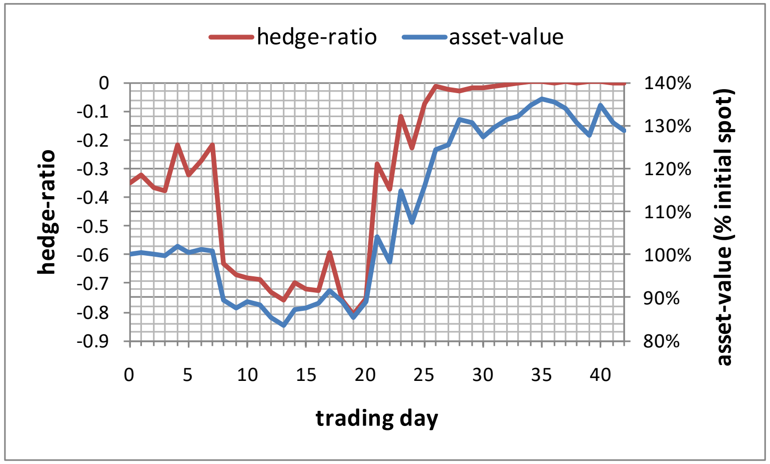
The P&L outcome is of course uncertain, and can be far less favorable if the underlying moves sharply, as shown next.



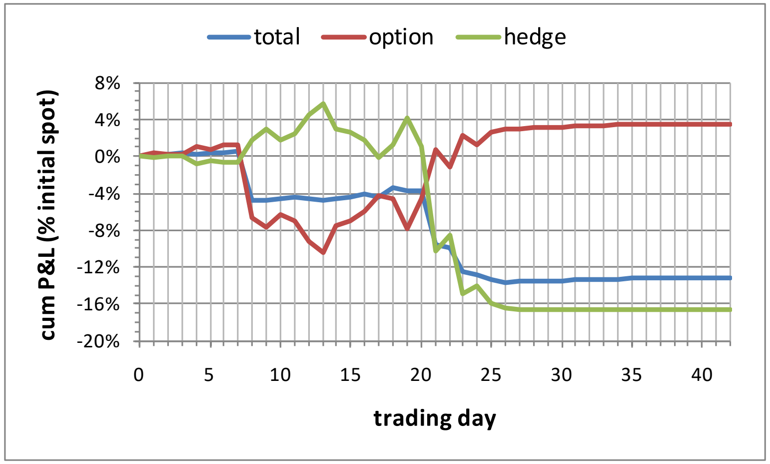


HEDGE PERFORMANCE

Tail loss scenario (1 yr 99.9 confidence level) total P&L sample path hedge performance from OHMC analysis of sell-hedge 42 day 95% strike put in high volatility regime (hedge mode e)



While the hedging strategy seeks to maintain an *expected* P&L over a hedging interval such that the Sortino-Ratio is 1 the total P&L outcome is negative.



Due to the sudden drop in the asset around the 7th day the losses incurred on the sell option position are greater in magnitude than the gains from the hedge position in the underlying. In the sharp upswing in the asset near the 20th trading day, the gains arising from the sell option position are less than the losses incurred due to the hedge position.

Such steep losses incurred by the derivative seller-hedger outline the need for risk-capital by the seller-hedger.

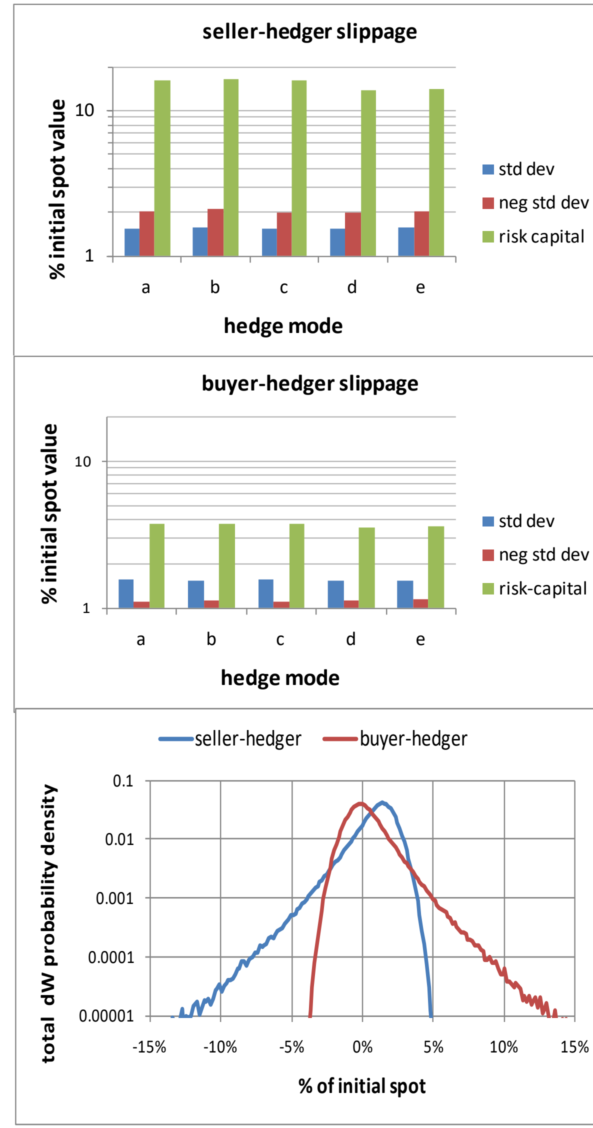
ASYMMETRY OF RESIDUAL RISKS

Hedge slippage measures for a 42 day 95% strike put in high volatility regime (hedge mode e)

The hedging strategy includes local risk-premium constraints on the expected change in wealth, in addition to considering transaction costs and conditioning on 10-day realized volatility.

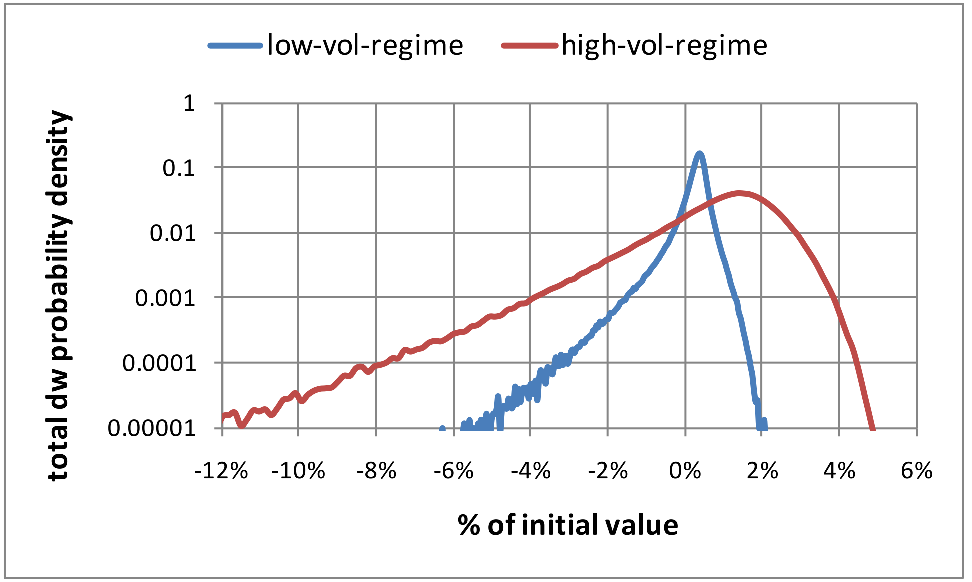
The positive modal wealth change for the option seller-hedger is also accompanied by a punishing loss-tail, and limited upside. Conversely, the option buyer-hedger's wealth change distribution has a modal wealth change value that is slightly negative, but is also accompanied by a limited loss-tail, and larger potential gains.

Understanding these asymmetries is central to developing successful investment & trading strategies involving options.



VOLATILITY REGIME

Information about the volatility regime enters into our analysis through the GARAM simulations of the asset returns. OHMC further propagates that information into hedge performance assessment.



Comparison of option sell-hedge total wealth change distribution in low and high volatility regime, for 42 day 95% strike put (**hedge mode e).** The hedging strategy imposed the same risk-premium constraints at each hedge time step (Sortino-Ratio = 1). The resultant total wealth change distributions are widely different, owing to different regimes of realized volatility.

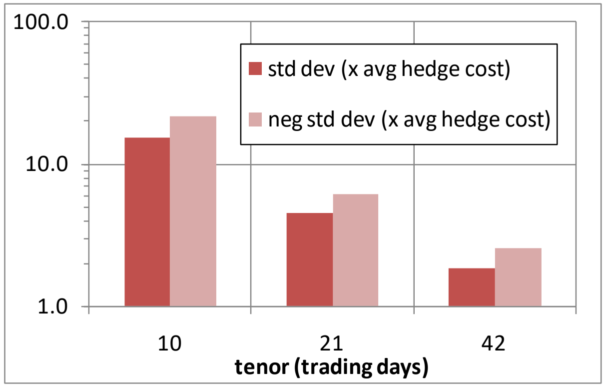
OHMC helps define trading strategies that are explicitly informed of such differences. For example, to limit the volatility of a option trading strategy, the *gearing* of the trading strategy needs to respond to changing regimes.

TERM DEPENDENCE

Term-dependence of residual risk for a seller-hedger of a 95% strike put (hedge mode a)

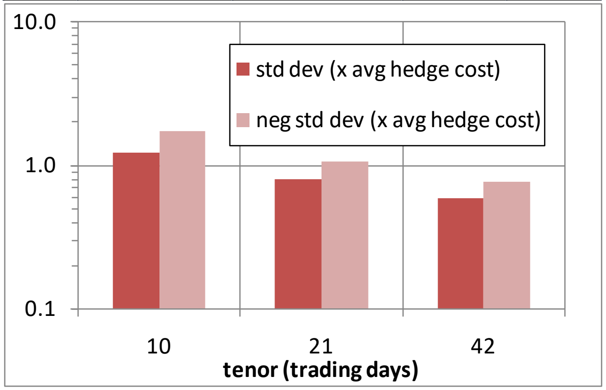
Low-Volatility Regime

tenor (days)	std dev (x avg hedge cost)	neg std dev (x avg hedge cost)	skewness	kurtosis
10	15.5	21.6	-23	1061
21	4.5	6.1	-6.1	113
42	1.85	2.56	-3.8	42



High-Volatility Regime

tenor (days)	std dev (x avg hedge cost)	neg std dev (x avg hedge cost)	skewness	kurtosis
10	1.23	1.72	-3.8	38
21	0.79	1.07	-2.9	30
42	0.59	0.77	-2.4	18



- The shorter the option tenor is, the greater the hedging errors are compared to the average hedge cost

- As longer options typically incur average higher hedging costs, the hedge-slippage associated with sharp moves in the underlying, that can be particularly troublesome to hedge if the option expiry is imminent, becomes a smaller fraction of the average hedging costs

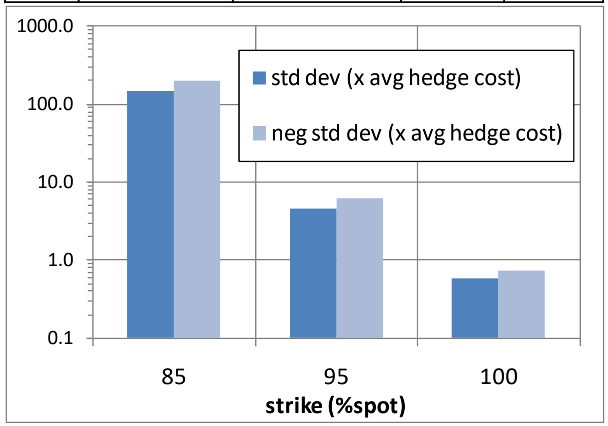
- Low-volatility regimes (at inception) exhibit greater treachery of tail-risks!

STRIKE DEPENDENCE

Strike dependence of residual risk for a seller-hedger of a 21 day put (hedge mode-a)

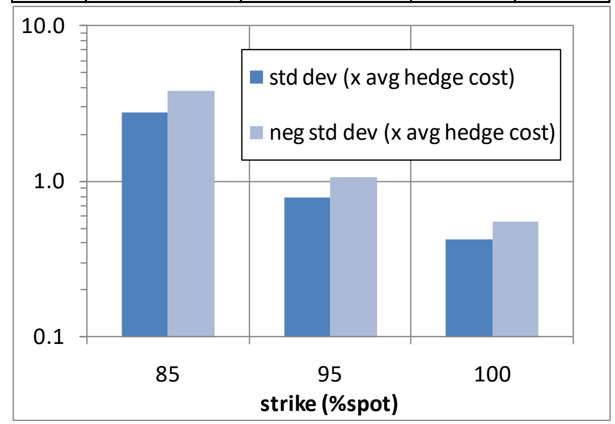
Low-Volatility Regime

strike (% spot)	std dev (x avg hedge cost)	neg std dev (x avg hedge cost)	skewness	kurtosis
85	147	200	-168	41086
95	4.5	6.1	-6.1	113
100	0.58	0.74	-2.3	19



High-Volatility Regime

strike	std dev (x avg hedge cost)	neg std dev (x avg hedge cost)	skewness	kurtosis
85	2.8	3.8	-5.5	91
95	0.79	1.1	-2.9	30
100	0.42	0.55	-2.4	21



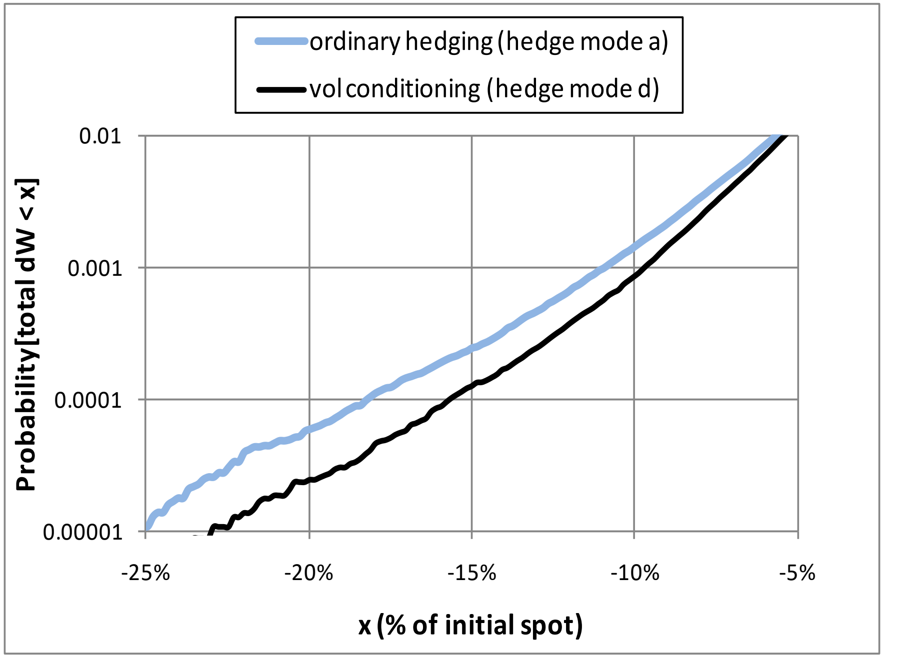
- Out of money options have a larger hedging error compared to average hedging costs than options with strikes closer to the spot asset value at inception

- While the body residual-risk measures are increasing multiples of the average hedging costs as one looks at strikes further out of money, the tail risks increase even more, relative to the average hedging costs

- Low-volatility regimes (at inception) exhibit greater treachery of tail-risks!

CONDITIONING ON REALIZED VOLATILITY

Impact of conditioning hedging and valuation on 10 day trailing realized volatility for a sell 95% strike 42 day put in a high volatility regime



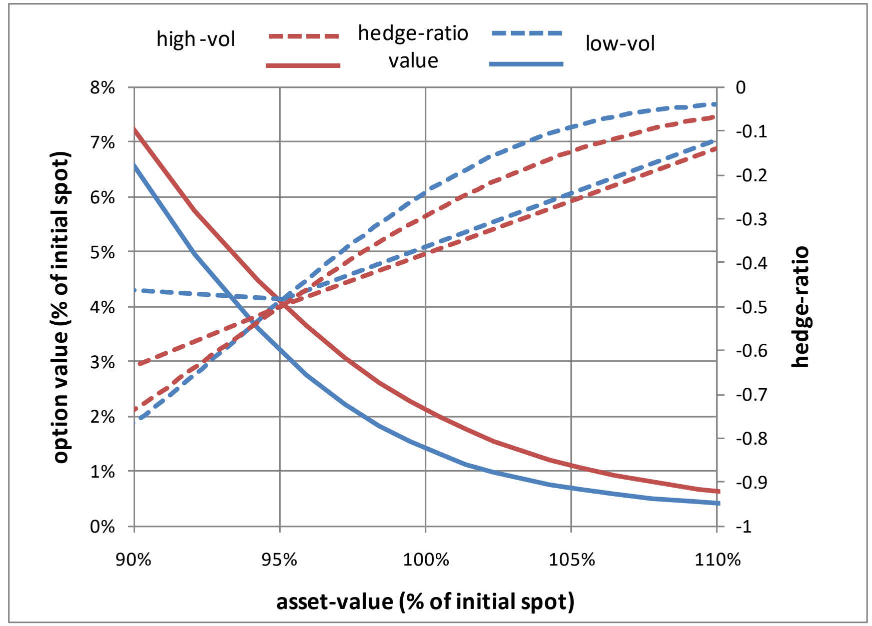
At high confidence levels, conditioning on trailing volatility shrinks the tail losses

This is due to the anticipatory nature of hedging while conditioning on realized volatility – owing its origin in the temporal persistence of the squared returns

CONDITIONING ON REALIZED VOLATILITY

Impact of conditioning hedging and valuation on 10 day trailing realized volatility for a sell 95% strike 42 day put in a high volatility regime

hedge mode d



The results of OHMC on the 21st day are depicted above. The 10-day realized volatility in the low-vol case is 17% and the 10-day realized volatility in the high-vol case is 32%. By accounting for such *forward realized volatility sensitivity* of the hedging strategy, OHMC with conditioning on realized volatility results in a trading strategy with thinner tail-losses. While accounting for the forward realized volatility dynamics yields benefits even for simple European style options, they can be even more pertinent for derivative contracts with embedded forward starting options (e.g., Cliquets).

FRAMEWORK FOR INFERRING OPTION RISK-PREMIA

Trading Calendar	$\{t_0 = 0, t_1, t_2, \ldots\ldots, t_k, t_{k+1}, \ldots\ldots, t_{K-1} = T\}$
Time t_a Value of Unit Cash Inflow at t_b	$df(t_a, t_b)$
Asset Value	$s(t_k)$
Asset Evolution Joint Density Function	$f_{s(t_1), s(t_2), \ldots, s(T)}(s(t_1), s(t_2), \ldots, s(T))$
Conditioning Variables	$\varpi_j(t_k);\ 1 \le j \le J$
Hedging Strategy	$\phi(s(t_k), \varpi_1(t_k), \varpi_2(t_k), \ldots\ldots, \varpi_J(t_k), t_k)$
Cost of Hedging Strategy	$C(t_0)$
Change in Wealth in Hedge Interval	$\Delta W_{t_j}(t_j, t_{j+1})$
Total Wealth Change	$\Delta W = \Delta W_{t_0}(t_0, T) = \sum_{j=0}^{K-2} \Delta W_{t_j}(t_j, t_{j+1}) df(t_0, t_j)$
Probability Density of Change in Wealth	$f_{\Delta W}(x)$
Average Change in Wealth	$\overline{\Delta W} = \int_{-\infty}^{\infty} \Delta W f_{\Delta W}(x) dx$
Actual Traded Option Price	$V(t_0)$
Option Seller's Average Change in Wealth	$\Psi = V(t_0) - C(t_0) + \overline{\Delta W}$

FRAMEWORK FOR INFERRING OPTION RISK-PREMIA

Risk-Capital at Confidence Level $p_S(T)$	$Q = \left\{\overline{\Delta W} - q\right\}^{+} \ni \text{Probability}\left\{\Delta W > q\right\} = p_S(T)$
Implied Return on Risk-Capital	$\Theta = \Psi / Q$
Implied Rate of Return on Risk-Capital	$\theta = (1/T)\ln(\Theta + 1)$

Standard Deviation of Wealth Change	$(\sigma_{\Delta W})^2 = \int_{-\infty}^{\infty} \left(x - \overline{\Delta W}\right)^2 f_{\Delta W}(x)dx$
Implied Sharpe-Ratio	$\Lambda = (\Psi / \sigma_{\Delta W})\sqrt{1/T}$

Negative Semi-Deviation of Wealth Change:	$\left(\sigma_{\Delta W}^{-}\right)^2 = \dfrac{\int_{-\infty}^{\infty} \min\left(x - \overline{\Delta W}, 0\right)^2 f_{\Delta W}(x)dx}{\int_{-\infty}^{\overline{\Delta W}} f_{\Delta W}(x)dx}$
Implied Sortino-Ratio:	$\Lambda^{-} = \left(\Psi / \sigma_{\Delta W}^{-}\right)\sqrt{1/T}$

Positive Semi-Deviation of Wealth Change:	$\left(\sigma_{\Delta W}^{+}\right)^2 = \dfrac{\int_{-\infty}^{\infty} \max\left(0, x - \overline{\Delta W}\right)^2 f_{\Delta W}(x)dx}{\int_{\overline{\Delta W}}^{\infty} f_{\Delta W}(x)dx}$
Implied Artemis-Ratio:	$\Lambda^{+} = \left(\Psi / \sigma_{\Delta W}^{+}\right)\sqrt{1/T}$

EXAMPLE OPTION RISK-PREMIA INFERENCE

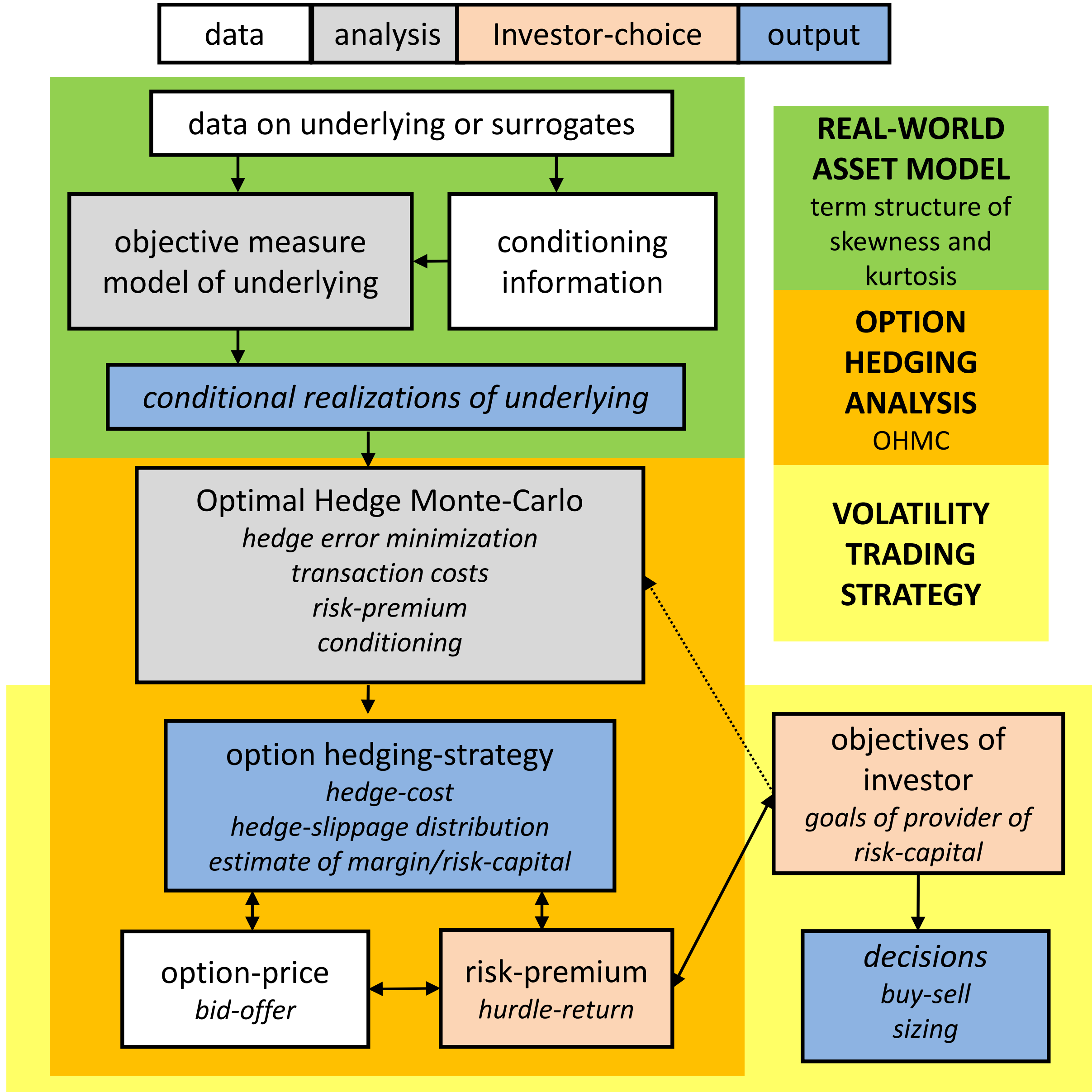
strike	avg-hedge-cost	hedge-slip stdev	hedge-slip stdevneg	1 yr 99.9% risk-capital	hedge-ratio (%)	BID	ASK	seller-hedger expected P&L	buyer-hedger expected P&L	seller-hedger Sortino Ratio	buyer-hedger Artemis Ratio
1135	25.86	11.67	15.04	164.6	-49.65	33.0	35.0	7.14	-9.14	1.63	-2.09
1100	12.91	10.04	13.42	152.2	-31.52	22.1	23.7	9.19	-10.79	2.35	-2.76
1065	6.03	7.78	10.81	134.7	-18.02	15.1	16.5	9.07	-10.47	2.88	-3.32
1030	2.71	5.98	8.34	116.5	-9.60	10.4	11.8	7.69	-9.09	3.16	-3.74
995	1.20	4.66	6.34	112.8	-4.92	7.0	8.6	5.80	-7.40	3.14	-4.00
960	0.53	3.59	4.88	117.0	-2.45	4.8	6.2	4.27	-5.67	3.01	-3.99
925	0.23	2.69	3.71	115.8	-1.20	3.1	4.5	2.87	-4.27	2.65	-3.94
890	0.10	1.94	2.73	100.7	-0.58	2.1	3.4	2.00	-3.30	2.51	-4.15
855	0.04	1.35	1.93	77.9	-0.27	1.6	2.5	1.56	-2.46	2.77	-4.37
820	0.02	0.92	1.33	47.7	-0.12	1.1	2.0	1.03	-1.93	2.67	-5.00
785	0.008	0.63	0.90	14.8	-0.06	0.7	1.6	0.64	-1.54	2.44	-5.85
750	0.003	0.44	0.64	2.1	-0.02	0.3	1.0	0.30	-1.00	1.60	-5.37

Recent sample calculations for short dated listed SPX PUT options

- In general the listed SPX PUT options are axed toward the seller-hedger…….
- Risk-premiums vary, driven by demand and supply and realized volatility…
- Occasionally SPX PUT options may be axed towards the buyer-hedger……

GARAM-OHMC BASED VOLATILITY TRADING SYSTEM

- Given an option price a buy-sell decision is made by inferring its risk-premium, and investor risk tolerance and goals, accounting for market conditions
- Underlying is traded to hedge portfolio of traded options
- Absolute portfolio risk profile is maintained to limit large losses
- Conditioning hedging on attributes of underlying in addition to spot value helps control risks



Optimal Dynamic Hedging of Equity Options
Residual Risks, Transaction Costs & Conditioning

Vivek Kapoor

vivek2.kapoor@citi.com
Citi

Global Derivatives Trading & Risk Management
May 2010