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Research Article

Arnold's Digitized Summation Technique and Generalized Notion of the Collatz Conjecture

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Abstract

Many mathematicians have studied Collatz conjecture and its applications; however, it remains an open problem in the field of number theory and is interesting to study. In the present paper, we give a generalized notion of Collatz conjecture as per the new notion of Arnold's Digitized Summation Technique which involves adding digits of a number until we are left with only one single digit. Moreover, a detailed description of the first twenty positive integers is given.

Keywords: Positive integers; Arnold's digitized summation technique; Digitized number form; Collatz Conjecture; Hailstone sequence.

Introduction

Collatz conjecture has been studied in different fields including topological spaces [1-2].The Collatz Conjecture still remains an open problem in the field of number theory [3-4]. Even after 82 years, explaining whether all the positive integers in the Collatz sequence eventually reach the trivial 4, 2, 1 cycle is still extremely difficult. From Paul Erdős famous statement, "Mathematics may not be ready for such problems," to Jeffery Lagarias' own point of view, "This is an extraordinarily difficult problem, completely out of reach of present day mathematics," just shows the stature of the problem. However, comprehending the conjecture is very simple [5-10]. When it is an odd number, you multiply it by 3 and add one [11]. If it is an even number, you divide it by 2. When you repeat this process, you will eventually end up with the trivial 4, 2, 1 cycle. In this paper we are going to look at the function f (n) = n/2 if n≡ 0 (mod 2), 3n+1 if n≡1 (mod 2). Whether this holds true for all the positive integers is still not proven. A counterexample would either have a different cycle from the trivial 4, 2, 1 cycle or has a divergent trajectory leading to infinity. By 2017 $87*2^{60}$ of all the starting numbers which have been tested eventually lead to the trivial 4, 2, 1 cycle.

We are now going to look at a very new idea known as Arnold's Digitized Summation Technique due to Arnold Okoth (The first author of this paper). This is a process where you keep adding the digits of a number until you are left with only one single digit. That remaining digit is known as the digitized form of that number. This form of looking at numbers was commonly used by renowned scientist and mathematician Nikola Tesla. It is a simple concept with astonishing results when it comes to its implications to the field of number theory. With the aid of the ADST Collatz cycle we will be able to see how all numbers eventually end up at the trivial 4, 2, 1. The Collatz Conjecture describes the iterations of integers applied to a very simple function. The conjecture specifically states: "Starting from any positive integer n, iterations of the function $C(x)$ will eventually reach the number 1. Thereafter iterations will cycle taking successive values 4, 2, 1,...". To define a basic term, an integer x will be defined as odd when $x \equiv 1 \pmod 2$. Likewise, x will be defined as even when $x \equiv 0 \pmod 2$. With those

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common terms specified, the following is the function known as the Collatz function (Eq. (q)).

$$C(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd} \\ \frac{1}{2} * x & \text{if } x \text{ is even} \end{cases} \quad (1)$$

The Collatz function is named as such with respect to its originator [12-15]. However, for the purpose of analysis, a more succinct function describes the same graph with less iteration, as the odd component of the function, $C(x) = 3x+1$, ensures that the following iteration will result in an even value. The allure of the problem and frustration of many mathematicians is the seemingly predictable randomness of the iterations. The specific number of iterations it takes for a starting value to reach 1 is referred to as the "total stopping time." The total stopping time is a very important value, as it is the point of focus for much of the research done on the Conjecture [16-18]. As an example of this tantalizing randomness, the recorded total stopping time of initial values that are relatively close to one another seem to form patterns, but in the end are random. Often two or three initial values in a row have the same stopping time, but in a completely unpredictable way. The initial values 1004, 1005, and 1006 all have total stopping times of 45, for example. Also, initial values in the range of 1000-1099, only nineteen total stopping times exist, with the total stopping times of 23 and 80 appearing 17 and 16 times respectively. Tendencies of the total stopping time can be loosely mapped, but ever so loosely. Iteration cycles have been studied at depth, but no avenue of research has proved fruitful in the search of a proof. While the problem itself remains unsolved and seemingly unapproachable, a fair amount of research has been done on the generalizations of the problem when viewed as a specific case of a more general class of functions. Some of these more general functions are analyzable. Such generalized $3x+1$ problems include the "3x+d" problem which showed that all integer orbits are eventually periodic for $d \geq -1$ [19], and the "qx+1" problem which showed that problems of similar structure can indeed be proven [20].

These results provide a plausible model for the specific $3x+1$ problem, but do not necessarily approach a solution. There are a many factors that contribute to the overall difficulty of the problem. Pseudorandomness, one of a few major influences on the difficulty and elusiveness of the Collatz Conjecture, is related to ergodic theory which is beyond the technical scope of this overview. However, according to [21], the connection shows that "the iterates of the shift function are completely unpredictable in the ergodic theory sense". This pseudorandomness can be observed for all values of x until $x = 2^n$ for any positive integer n. "This supports the $3x+1$ conjecture and at the same time deprives us of any obvious mechanism to prove it, since mathematical arguments exploit the existence of structure, rather than its absence." Another issue, which is described in depth in [22] "Unpredictable Iterations", deals with the inability of the any sort of computer generated algorithm to predict nearly anything about the iterations in the long run. This roadblock which Conway refers to as "non-computability" reveals that the problem could indeed be unsolvable, and a method to approach the issue is [23].

From the perspective of an individual less applauded in the field, Peter Schorer of Hewlett-Packard Laboratories claims that "one reason the problem is so difficult is that (informally) the structure of counterexamples to the $3x+1$ Conjecture, and the structure of non-counterexamples, are so similar. For example, the inverse of each range element y of the $3x+1$ function, be that range element a counterexample or a non-counterexample, is an infinitary tree with y as a root. Furthermore, all the properties of these trees that we are aware of, are the same regardless whether the root is a counterexample or a non-counterexample." Schorer then proceeds to attempt to prove the conjecture by showing that there is no difference between counterexample tuples and non-counterexample tuples. His proof has yet to gain any wide acceptance. The known difficulty of the problem as well as the seemingly simple nature of the function has lead to research in many different fields, namely number theory, dynamical systems, computer science, ergodic theory, probability theory, and computational theory.

In [5] they worked on the problem from a number theory perspective on the connection that the problem is arithmetic in nature. Classes of generalized versions on the function have been defined under certain conditions. In dynamical systems, the problem is studied via the behavior of the function under iteration. Computational and Fractran models have been used to show the validity of the conjecture to a very large scale in the computer science field. Ergodic theory deals with the presence of an invariant measure in the dynamic system. Probability theory attempts to model the behavior of the iteration. Lastly, computational theory connects with the Collatz Conjecture via [23], who states that "there is a generalized $3x+1$ function whose iteration can simulate a universal computer". [5] The fact that the Collatz Conjecture spreads across so many different fields of mathematics has allowed many great minds to work on and contribute to the knowledge base of the problem. It has opened up avenues of research in all of these disciplines and has leads to some important results outside of the conjecture itself. To the avail of many a mathematician, in spite of the results generated by supercomputers, and mocking the analysis of generalized forms of the function, the Collatz Conjecture remains unsolved and seems to be unsolvable. A fair amount is known about it, but the vast majority of that knowledge has proved useless in the realm of proving the conjecture. The broad scope of the problem, its seemingly simple nature, and the vast depth of related problems will continue to intrigue and puzzle mathematicians for, quite possibly, a very long time to come.

Research methodology

In this section, we give the definitions, examples and techniques which are useful as the research methodology.

Definition 2.1

Arnold's Digitized Summation Technique (ADST): Refers to adding the digits of a number until you are left with only one single digit.

Definition 2.2

Digitized number form: Refers to the digit we obtain after applying ADST to a number.

Definition 2.3

Hailstone sequence: Refers to the sequence of descending and ascending numbers which you obtain when you perform the Collatz process. The ascension is caused by the 3n+1 operation while the descending aspect is brought by the n÷2 operation.

Definition 2.4

Arnold’s values: These refer to values assigned to numbers based on their digitized number form. We basically we have only 3 signs based on the multiples of 3. If a number is 3 or a multiple of 3 it gets a value of (0), if the number is exactly before 3 or a multiple of 3 it is given a (–) value and finally, if a number is exactly after 3 or a multiple of 3 it gets a (+) value.

Definition 2.5

Stopping time: In the Collatz sequence you will notice that numbers eventually lead to the trivial 4, 2, 1 cycle which repeats itself. Therefore, 1 is regarded as the stopping time when performing the Collatz process.

Results and discussion

In this section, we give the results of our study and their discussions. We begin by the following fundamental result.

Theorem 3.1.

If p_0 is such that $p_0 \equiv -1 \pmod{2^n}$, where n is the largest integer such that this congruence holds, then $\varphi(p_0) = n$.

Proof.

Although this theorem follows from (1) by observing that n+1 ≡ 0 ($mod\ 2^k$), k-maximal implies that $\alpha^k(n)$ is even and for 1 ™ $i < k$, $\alpha^i(n)$ is odd, we offer an alternative proof: Suppose without loss of generality that p_0 is odd. Then $3p_0 + 1 \equiv -2\ (mod\ 2^n)$. Repeating this argument we get that $p_2 \equiv -1\ (mod\ 2^{n-2})$ and $p_{n-1} \equiv -1\ (mod\ 2)$. Since all those p_j are odd, this gives us that $\varphi(p_0)=n$. Suppose now that $\varphi(p_0) > n$, i.e. $p_n \equiv -1\ (mod\ 2)$. But $2p_n - 1 = 3p_{n-1}$, and since $2p_n \equiv -2\ (mod\ 2^2)$. So, $p_0= \frac{2pn - 1}{} \equiv -1\ (mod\ 2^2)$. Repeating this reasoning we get that $p_0 \equiv -1\ (mod\ 2^{n+1})$, which contradicts the fact that n is maximal.

Corolary 3.2

$\varphi(m)$ is finite for every m.

Corolary 3.3

For every natural number k there are infinitely many numbers n such that $\varphi(n) = k$.

Proof

Just take $n = l \cdot 2^k - 1$, where l is an odd number. From the formula (1) it is easy to see that $\alpha^k(n)$ 2 (mod 3) for all k such that this number is an integer. In particular $\alpha^{\varphi(n)}(n)$ 2 (mod 3), but since (from the definition of $\varphi(n)$) it is also even, we have that $\alpha^{\varphi(n)}(n) \equiv 2\ (mod\ 6)$. But now, since every even number is eventually taken to an odd number by successive applications of the function T , and T executes the operation α on an odd number m exactly $\varphi(m)$ times, we deduce that every number is taken to a number congruent to 2 (mod 6).

Theorem 3.4

In order to prove the Collatz conjecture, it is sufficient to prove it for every number congruent to 2 (mod 6).

Proof

Since $\varphi(m)$ is finite for every integer m it is not possible for an unbounded trajectory to consist entirely of odd numbers and thus our initial upper bound can be improved. From Theorem 1 we conclude that if m ™ $m_0 = 2^k$ 1 for some integer k, then $\varphi(m)$ ™ $k = \log_2(m_0 + 1)$. After $\varphi(m)$ applications of α we must divide the result by 2 at least once. Since our goal is an upper bound, we will assume division by 2 occurs exactly once and that this process continues indefinitely. Since the Collatz conjecture states that when you have an odd number you multiply it by 3 and add one then if it is an even number, you divide it by 2. When you repeat this process, you will eventually end up with the 4, 2, 1 cycle. Examples include:

n=20 we get the sequence 20,10,5,16,8,4,2,1,(4,2,1,4,2,1,…)
n=19 we get the sequence 19,58,29,88,44,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1,(4,2,1,4,2,1,…)
n=18 we get the sequence 18,9,28,14,7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1,(4,2,1,4,2,1,…)
n=17 we get the sequence 17,52,26,13,40,20,10,5,16,8,4,2,1,(4,2,1,4,2,1,)
n=16 we get the sequence 16,8,4,2,1,(4,2,1,4,2,1,…)
n=15 we get the sequence 15,46,23,70,35,106,53,160,80,40,20,10,5,16,8,4,2,1,(4,2,1,4,2,1,…)
n=14 we get the sequence 14,7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1,(4,2,1,4,2,1,…)
n=13 we get the sequence 13,40,20,10,5,16,8,4,2,1,(4,2,1,4,2,1,…)
n=12 we get the sequence 12,6,3,10,5,16,8,4,2,1,(4,2,1,4,2,1…)
n=11 we get the sequence 11,34,17,52,26,13,40,20,10,5,16,8,4,2,1,(4,2,1,4,2,1,…)
n=10 we get the sequence 10,5,16,8,4,2,1,(4,2,1,4,2,1,…)
n=9 we get the sequence 9,28,14,7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1,(4,2,1,4,2,1,…)
n=8 we get the sequence 8,4,2,1,(4,2,1,4,2,1,…)
n=7 we get the sequence 7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1,(4,2,1,4,2,1,…)

n=6 we get the sequence 6,3,10,5,16,8,4,2,1,(4,2,1,4,2,1…)
n=5 we get the sequence 5,16,8,4,2,1,(4,2,1,4,2,1,…)
n=4 we get the sequence 4,2,1,(4,2,1,4,2,1,…)
n=3 we get the sequence 3,10,5,16,8,4,2,1,(4,2,1,4,2,1…)
n=2 we get the sequence 2,1,(4,2,1,4,2,1,…)
n=1 we get the sequence 1,(4,2,1,4,2,1,…)

Next, we want to consider Collatz Conjecture using ADST. The Collatz conjecture seems to be forming a different sequence for numbers which are not multiples. The sequence for n=4 is different from the sequence of n=3. Before we look at the ADST and its implications to the Collatz Conjecture, let us first look at an in depth explanation of the ADST and why it is important when it comes to analyzing the Collatz Conjecture. Digitized number form: It refers to the digit we obtain after applying ADST to a number.

Example 3.5

For 2345 we have 2+3+4+5 = 14 1+4=**5.**

Therefore 5 is the digitized number form of 2345 while the entire process is also known as ADST.

More examples include:

a. 145 1+4+5=10 1+0=1.
The digitized number form of 145 is 1.
b. 3854 3+8+5+4=20 2+0=2.
The digitized number form of 3854 is 2.
c. 3000 3+0+0+0=3
The digitized number form of 3000 is 3.
d. 22945 2+2+9+4+5=22 2+2=4
The digitized number form of 22945 is 4
e. 275 2+7+5=14 1+4=5.
The digitized number form of 275 is 5.
f. 6315 6+3+1+5=15 1+5=6
The digitized number form of 6315 is 6
g. 43 4+3=7
The digitized number form of 43 is 7
h. 53756 5+3+7+5+6=26 2+6=8.
The digitized number form of 53756 is 8.
i. 9
The digitized number form of 9 is 9.

By considering table 1, when we look at numbers through ADST you will notice that among all the numbers from 1 till infinity we only have 9 possible digitized number forms. This is because we have only 9 digits in mathematics, excluding 0. In the table 1 you will observe that the first column is named AV. which stands for Arnold's Values. These values help in analyzing the numbers using ADST as explained in depth with the analysis of prime numbers. Basically, we have only 3 signs based on the multiples of 3. If a number is 3 or a multiple of 3 it gets a sign of (0), if the number is exactly before 3 or a multiple of 3 it is given a (–) sign and finally, if a number is exactly after 3 or a multiple of 3 it gets a (+) sign. Next we consider Collatz trees. There are various Collatz trees which have been formulated by computer programs showing the Collatz sequence. Some of the Collatz trees are illustrated in Fig. 1 and 2.

Table 1. Digitized number forms and Arnold’s values

AV.	+	-	0	+	-	0	+	-	0	+	-	0	+	-	0	+	-	0	+	-
Integers	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
DNF	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	1	2

The Collatz trees in Fig. 1 and 2 both form unpredictable patterns. We also find that the Collatz trees can become very large in size as the values of numbers increase. It would therefore be unidealistic to represent a Collatz tree with numbers up to 10^{10}.

ADST Collatz cycle

Unlike the Collatz tree, the ADST Collatz cycle can be used to represent numbers from 1 till infinity. Numbers portray a particular property which is hidden and can only be visible when you observe them in their digitized number form. Notice what happens when we observe the sequence below using their digitized form. The sequence of powers of 2 appear as follows: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072. When you look at the digitized form of the above sequence it will look as follows; 1, 2, 4, 8, 7, 5, 1, 2, 4, 8, 7, 5, 1, 2, 4, 8, 7, 5. With the digitized form you will notice that the sequence 1, 2, 4, 8, 7, 5 repeats itself.

Interestingly, we also observe that the trivial 4, 2, 1 cycle does not just appear at the end of the Collatz sequence, but also throughout the Collatz sequence. This is only visible when you look at the sequence through ADST. This sequence will form the basis of our ADST Collatz cycle.

This is because part of the Collatz conjecture involves dividing even numbers by 2. The n/2 operation forms the descending aspect of the hailstone sequence. If you only consider the odd numbers in the sequence generated by the Collatz process, then each odd number is on average ¾ of the previous one. Therefore, the geometric mean of the ratios outcomes is ¾. This yields a heuristic argument that every Hailstone sequence should decrease in the long run, this is not evidence against other cycles, but against divergence. The geometric and heuristic arguments are not a proof as it assumes the Hailstone sequence is assembled from uncorrelated probabilistic events. However, the ADST Collatz cycle not only shows the evidence against other cycles but how the hailstone sequence are assembled. This is illustrated by Fig. 1. The cycle in Fig. 2 represents the ADST Collatz cycle.

The arrows represent the n÷2 operation which implies that the numbers outside the brackets are digitized number form of even numbers while the numbers inside the brackets refers to the digitized number form of odd numbers. The arrows do not apply for the numbers that are inside the brackets. If you get an odd number (whose digitized number forms are inside the brackets) and you perform the 3n +1 operation, you will get an even number whose digitized number form is outside the brackets but inside the same circle.

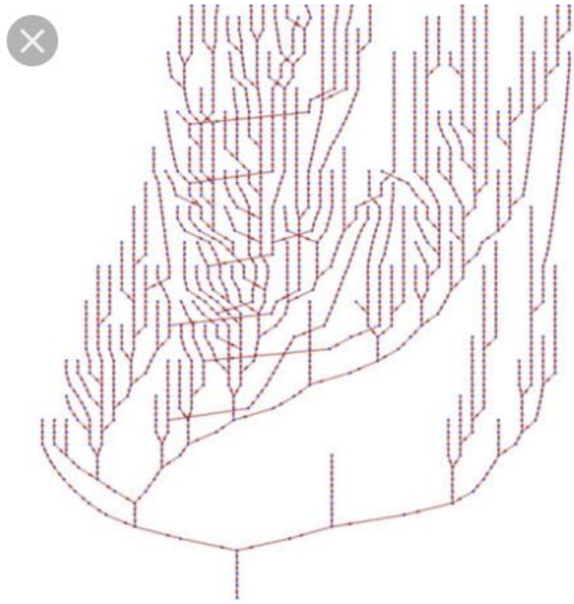

Fig. 1. Collatz tree 1 [5]

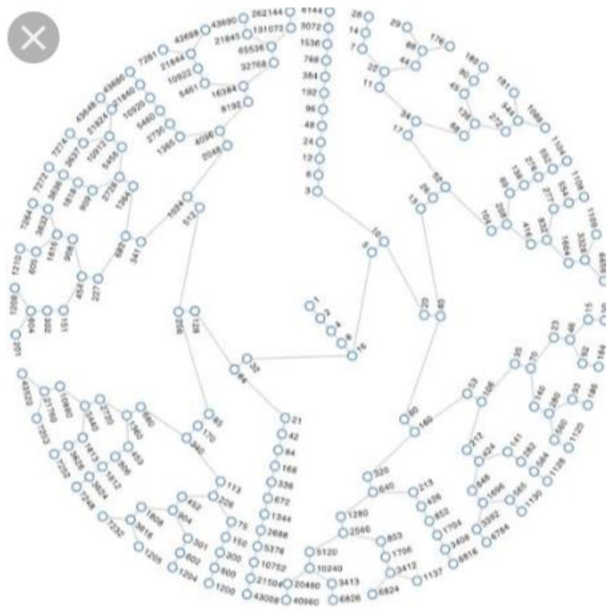

Fig. 2. Collatz tree 2 [23]

However, when you divide even numbers which are multiples of 6 and 9 you will obtain a different cycle from the 1, 2, 4, 8, 7, 5 sequence which is used for the ADST cycle. For multiples of 6 you will obtain a repeating 6, 3 cycle. For multiples of 9 you will obtain a repeating 9, 9 cycle as observed in Fig. 3. The 3x + 1 operation eliminates the 6, 3 cycle and the 9, 9 cycle since the addition in the operation implies that the result will not be a multiple of 3. The multiples 6 and 9 are also multiples of 3. You will realize from Fig. 3 that when you divide a number whose digitized number form is 3 or 6. The sequence will tend to rotate between 3 and 6 while if a number has a digitized number of 9 and you divide by 2 the sequence will tend to rotate at 9. The stopping time of the Collatz Conjecture (1) and the brackets are also positioned where the digitized number form has +ve Arnold's value.

Theorem 3.6.

For all powers of two, with odd k, it is $2^k \pmod 3 = 2\ \forall k \in \mathbb{N}\ |(k) \pmod 2 = 1$.

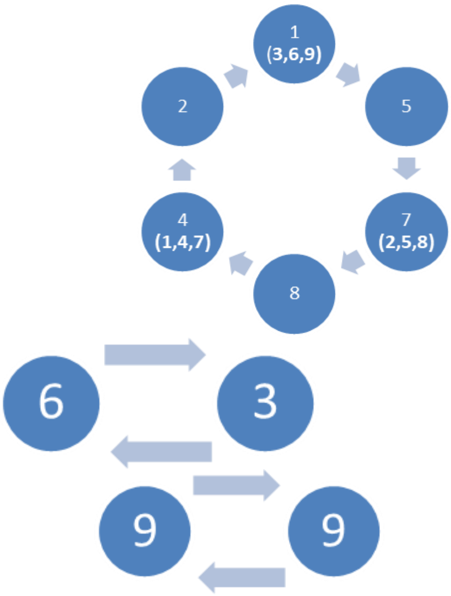

Fig. 3. ADST Collatz cycles

Conclusions

In the present work, we have considered the ADST Collatz cycle to the Collatz tree, we can be able to conclude that no matter which number you have, when you divide it by 2 you will always result with the same trivial cycle (4, 2, 1). We have given a generalized notion of Collatz conjecture as per the new notion of Arnold's Digitized Summation Technique which involves adding digits of a number until we are left with only one single digit.

Conflicts of interest

Authors declare no conflict of interest.

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