Math 3331 – ODE's

So far $y' = F(x,y)$

(1) <u>Separable</u> $\dfrac{dy}{dx} = f(x)\,g(y)$

separate $\dfrac{dy}{g(y)} = f(x)\,dx$ and integrate

(2) <u>Linear</u> $\dfrac{dy}{dx} + p(x)y = q(x)$

integrating factor $\mu = e^{\int p(x)dx}$

so $\dfrac{d}{dx}(\mu y) = \mu q$ ← sep & integrate

(3) <u>Bernoulli</u>

$\dfrac{dy}{dx} + p(x)y = q(x)y^n$ divide by y^n

$\dfrac{1}{y^n}\dfrac{dy}{dx} + p(x)\dfrac{y}{y^n} = q(x)$

let $u = \dfrac{y}{y^n}$ turns ODE into linear

Consider

$$\frac{dy}{dx} = y/x \quad \text{sep}$$

$$\frac{dy}{dx} = \frac{y^2}{x^2} \quad \text{sep}$$

$$\frac{dy}{dx} = \frac{y}{x} + 1 \quad \text{linear}$$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y^3}{x^2} \quad \text{Ber}$$

Now $\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x}$ neither sep, lin or Ber.

this type of ODE is called <u>Ricatti</u>

In general

$$\frac{dy}{dx} = a(x) + b(x)y + c(x)y^2$$

In order to solve we will need 1 sol^n (a guess) call this $y = y_1$

then let $y = y_1 + u$

Sub $y' = y_1' + u'$

So $y_1' + u' = a(x) + b(x)(y_1+u) + c(x)(y_1+u)^2$

$\underline{y_1'} + u' = \underline{a} + \underline{by_1} + bu + \underline{cy_1^2} + 2cy_1u + cu^2$

Since y_1 is 1 sol^n the $y_1' = a + by_1 + cy_1^2$

so the underlined term cancels and we are left with

$$u' = (b+2cy_1)u + cu^2$$

this new ODE is Bernoulli & can be solved

Ex 1 $\quad y' = -1 + \frac{y}{x} + y^2/x^2$

1 sol^n $y = x$ so $y' = 1$ R.S. $-1 + \frac{x}{x} + \left(\frac{x}{x}\right)^2 = -1+1+1$
$= 1$ ✓

let $y = x + u$ $\quad y' = 1 + u'$

$$\cancel{1} + u' = -1 + \frac{(x+u)}{x} + \frac{(x+u)^2}{x^2}$$

$$= -1 + 1 + u/x + \frac{x^2+2xu+u^2}{x^2} = \cancel{-1} + \cancel{1} + \frac{u}{x} + \cancel{1} + \frac{2u}{x} + \frac{u^2}{x^2}$$

$$\frac{du}{dx} = \frac{3u}{x} + \frac{u^2}{x^2} \qquad B.$$

$$\frac{1}{u^2}\frac{du}{dx} - \frac{3}{x}\cdot\frac{1}{u} = \frac{1}{x^2}$$

let $v = \frac{1}{u}$ $\quad \frac{dv}{dx} = -\frac{1}{u^2}\frac{du}{dx}$

$$-\frac{dv}{dx} - \frac{3}{x}u = \frac{1}{x^2} \Rightarrow \frac{dv}{dx} + \frac{3}{x}u = \frac{-1}{x^2} \quad \text{SF lin}$$

$$\mu = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\frac{d}{dx}(x^3 u) = -\frac{x^3}{x^2} = -x$$

$$x^3 u = -\frac{1}{2}x^2 + c \quad \text{so} \quad \frac{x^3}{u} = c - \frac{x^2}{2} = \frac{2c - x^2}{2}$$

so $$u = \frac{2x^3}{2c - x^2}$$

$$y = x + u = x + \frac{2x^3}{2c - x^2} = \frac{2cx + x^3}{2c - x^2}$$

sol^n $$y = \frac{2cx + x^3}{2c - x^2}$$

Ex2

$$y' = e^x - y + \frac{y^2}{e^x}$$

1 sd^n $y = e^x$ so $y = e^x + u$ $\quad y' = e^x + u'$

$$e^x + u' = e^x - (e^x + u) + \frac{(e^x+u)^2}{e^x}$$

$$\cancel{e^x} + u' = \cancel{e^x} - \cancel{e^x} - u + \cancel{e^x} + 2u + \frac{u^2}{e^x}$$

$$u' = u + \frac{u^2}{e^x} \quad B$$

$$\frac{1}{u^2}\frac{du}{dx} - \frac{1}{u} = \frac{1}{e^x} \qquad v = \frac{1}{u} \quad \frac{dv}{dx} = -\frac{1}{u^2}\frac{du}{dx}$$

so $$-\frac{dv}{dx} - \cancel{\frac{1}{u}} = \frac{1}{e^x} \qquad \frac{dv}{dx} + v = -\frac{1}{e^x}$$

$$\mu = e^x \text{ so } \frac{d}{dx}(e^x v) = -1 \qquad e^x v = -x + c$$

so $$v = \frac{c-x}{e^x} = \frac{1}{u} \qquad u = \frac{e^x}{c-x}$$

sd^n $$y = e^x + \frac{e^x}{c-x}$$