Calculus 3 - Vector Fields

Early in this course, we introduced vector functions

$$\vec{r}(t) = < f(t), g(t) > .$$

For example

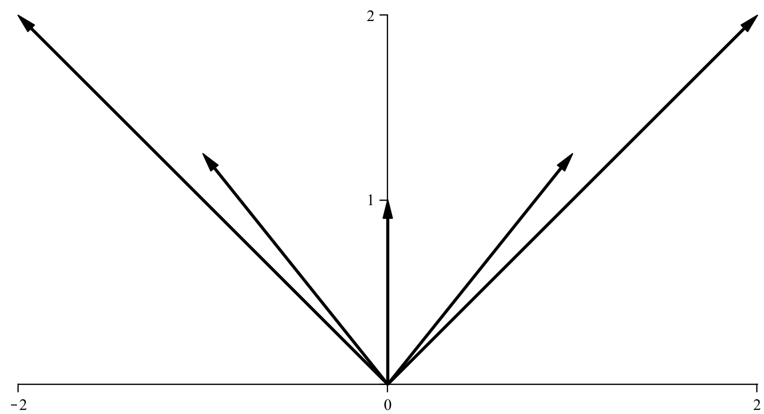


Figure 1: Vector Function $\vec{r}(t) = < t, \frac{1}{2}t^2 + 1 >$

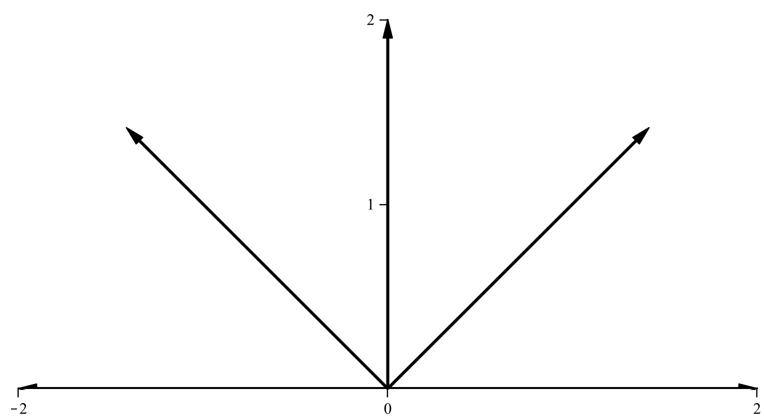


Figure 2: Vector Function $\vec{r}(t) = < \cos t, \sin t >$

We now introduce vectors that change with respect to position.

Consider the following

$$\vec{F}(x,y) =< -y, x > \tag{1}$$

We certainly could create a sort of table of values. So choosing ordered pairs (x, y) we see

$$\begin{array}{rr} \vec{F}(1,0) = < 0,1 > & \vec{F}(0,1) = < -1,0 > \\ \vec{F}(-1,0) = < 0,-1 > & \vec{F}(0,-1) = < 1,0 > \\ \vec{F}(1,1) = < -1,1 > & \vec{F}(-\frac{1}{2},\frac{1}{2}) = < -\frac{1}{2},-\frac{1}{2} > \end{array}$$

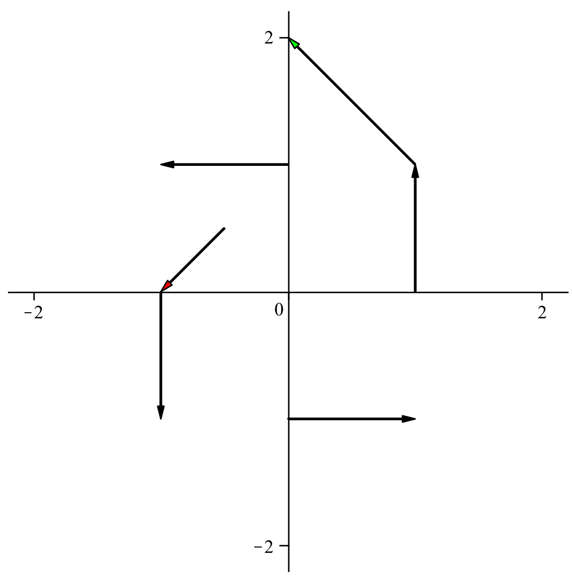


Figure 3: Vector Field $\vec{F}(x,y) =< -y, x >$

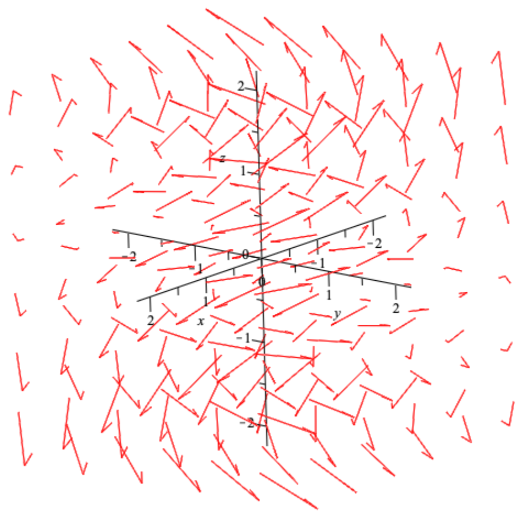
In general we have

$$\vec{F}(x,y) =< P(x,y), Q(x,y) > \tag{2}$$

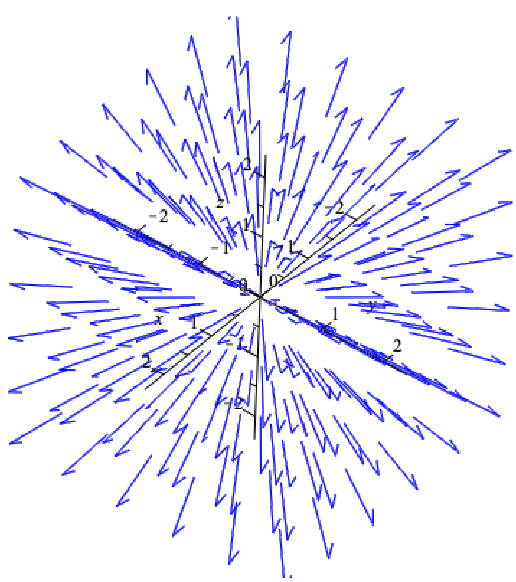
3D Vector Fields.

Consider for example

$$\vec{F}(x,y,z) = \langle -y, x, z \rangle, \tag{3}$$



$$\vec{F} = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle, \quad r = \sqrt{x^2+y^2+z^2} \tag{4}$$



Gradient Vector Fields

A vector field is called a *gradient* vector field (VF) if given some function $f(x, y)$ (called a potential) then

$$\vec{F} = \nabla f = \langle f_x, f_y \rangle \tag{5}$$

or in 3D if $f(x, y, z)$

$$\vec{F} = \nabla f = \langle f_x, f_y, f_z \rangle \tag{6}$$

For example, if $f(x, y) = x^2 + y^2$ then the gradient VF is

$$\vec{F} = \nabla f = \langle 2x, 2y \rangle \tag{7}$$

or if $f(x, y, z) = xy + z^2$ then the gradient VF is

$$\vec{F} = \nabla f = \langle y,\ x,\ 2z \rangle \tag{8}$$

Conservative Vector Field

A vector field is said to be *conservative* is a function f exists such that

$$\vec{F} = \nabla f. \tag{9}$$

So, for example one of these two vector fields is conservative

$$\vec{F} = \langle y,\ x \rangle, \quad \vec{F} = \langle -y, x \rangle \tag{10}$$

Test for Conservative Vector Fields

Let P and Q have continuous partial derivatives. A vector field

$$\vec{F} = \langle\, P(x,y), Q(x,y)\,\rangle \tag{11}$$

is conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{12}$$

Example 1. Is the following vector field conservative?

$$\vec{F} = \langle\, 2xy + 2,\; x^2 + 3y^2 \rangle \tag{13}$$

We first test that the VF is conservative. So

$$P = 2xy + 2, \quad Q = x^2 + 3y^2 \tag{14}$$

and

$$\frac{\partial P}{\partial y} = 2x, \quad \frac{\partial Q}{\partial x} = 2x \tag{15}$$

and they are the same, so yes, the VF is conservative.

Example 2. Is the following vector field conservative?

$$\vec{F} = \langle\, -y,\; x \rangle \tag{16}$$

We first test that the VF is conservative. So

$$P = -y, \quad Q = x \tag{17}$$

and since

$$\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1 \tag{18}$$

and they are not the same, so no, the VF is conservative.

Once we know that a VF is conservative, how do we find the potential f. Let us return to example 1. Here

$$\vec{F} = \langle\, 2xy + 2,\; x^2 + 3y^2 \rangle \tag{19}$$

Conservative vector fields are such that

$$\vec{F} = \nabla f = \langle\, f_x, f_y \,\rangle \tag{20}$$

so we equate the two so

$$\nabla f = \langle\, f_x, f_y \,\rangle = \langle\, 2xy + 2,\; x^2 + 3y^2 \rangle \tag{21}$$

or

$$\begin{aligned} f_x &= 2xy + 2, \\ f_y &= x^2 + 3y^2. \end{aligned} \tag{22}$$

Now we integrate each separately noting that integrating with respect to one variable, a function of integration is included.

$$f_x = 2xy + 2, \quad \Rightarrow \quad f = x^2 y + 2x + A(y), \tag{23a}$$

$$f_y = x^2 + 3y^2 \quad \Rightarrow \quad f = x^2 y + y^3 + B(x). \tag{23b}$$

As we want the same f in equation (23a) and (23b) we choose $A(y)$ and $B(x)$ accordingly. Thus,

$$A(y) = y^3 + c, \quad B(x) = 2x + c \tag{24}$$

(c is an arbitrary constant) and the potential is

$$f = x^2 y + 2x + y^3 + c. \tag{25}$$

Divergence and Curl of Vector Fields

We introduced the gradient of a function as

$$\nabla f = \left\langle\, f_x, f_y\, f_z \,\right\rangle, \tag{26}$$

we now introduce the *del* operator and define it as

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\, \frac{\partial}{\partial z} \right\rangle. \tag{27}$$

Given a vector field

$$\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle \tag{28}$$

we define the *Divergence* and *Curl* of a vector field.

Divergence $-$ We define divergence of a vector field as

$$\begin{aligned} \nabla \cdot \vec{F} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right\rangle \cdot \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle \\ &= P_x + Q_y + R_z. \end{aligned} \tag{29}$$

For example if

$$\vec{F} = \langle\, x^2, 2y, xyz \,\rangle \tag{30}$$

then

$$\nabla \cdot \vec{F} = 2x + 2 + xy. \tag{31}$$

Curl $-$ We define the curl of a vector field as

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix} \tag{32}$$

For example if $\vec{F} = \langle\, x^2, 2y, xyz \,\rangle$ then

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & 2y & xyz \end{vmatrix} \\ &= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y & xyz \end{vmatrix} \vec{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ x^2 & xyz \end{vmatrix} \vec{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ x^2 & 2y \end{vmatrix} \vec{k} \\ &= \langle\, xz, -yz, 0 \,\rangle \end{aligned} \tag{33}$$

In 3D, if a vector field is conservative, then

$$\nabla \times \vec{F} = \vec{0}.$$

Example 3

Show the following vector field is conservatives and find the potential f.

$$\vec{F} = \left\langle 2xz - \frac{3y}{x^2} + 1, \frac{3}{x}, x^2 + 4z \right\rangle \tag{34}$$

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz - \frac{3y}{x^2} + 1 & \frac{3}{x} & x^2 + 4z \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{3}{x} & x^2 + 4z \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2xz - \frac{3y}{x^2} + 1 & x^2 + 4z \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xz - \frac{3y}{x^2} + 1 & \frac{3}{x} \end{vmatrix} \vec{k} \\
&= \left\langle 0, -(2x - 2x), -\frac{3}{x^2} + \frac{3}{x^2} \right\rangle \\
&= \left\langle 0, 0, 0 \right\rangle
\end{aligned}$$

So the vector field is conservative. Thus, a potential f exists such that

$$\begin{aligned}
f_x &= 2xz - \frac{3y}{x^2} + 1, \\
f_y &= \frac{3}{x}, \\
f_z &= x^2 + 4z.
\end{aligned} \tag{35}$$

We integrate each

$$
\begin{aligned}
f &= x^2 z + \frac{3y}{x} + x + A(y, z), \\
f &= \frac{3y}{x} + B(x, z), \\
f &= x^2 z + 2z^2 + C(x, y)
\end{aligned}
\tag{36}
$$

Since we want a single f then choose

$$
A = 2z^2, \quad B = x^2 z + x, \quad C = \frac{3y}{x}, \tag{37}
$$

and so

$$
f = x^2 z + \frac{3y}{x} + 2z^2 + x + c. \tag{38}
$$