Math 1497 – Calc 2

Now we multiply vectors

Def^n If we have 2 vectors

$$\bar{u} = \langle u_1, u_2 \rangle \quad \& \quad \bar{v} = \langle v_1, v_2 \rangle$$

then $\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2$ (Dot Product)

Easily Extends to 3-D

$$\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Ex 1 $\bar{u} = \langle 1, 2 \rangle$ $\quad \bar{v} = \langle 5, -2 \rangle$

$$\bar{u} \cdot \bar{v} = (1)(5) + (2)(-2) = 5 - 4 = 1$$

Ex 2 $\bar{u} = \langle 3, 4, 0 \rangle$ $\quad \bar{v} = \langle 0, 4, 5 \rangle$

$$\bar{u} \cdot \bar{v} = (3)(0) + 4(4) + 0(5)$$

$$= 0 + 16 + 0 = 16$$

Properties Let $\bar{u}, \bar{v}, \bar{w}$ be vectors c scalar #

(1) $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$

(2) $\bar{u} \cdot (\bar{v} + \bar{w}) = \bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{w}$

(3) $c(\bar{u} \cdot \bar{v}) = (c\bar{u}) \cdot \bar{v} = \bar{u} \cdot (c\bar{v})$

(4) $\bar{0} \cdot \bar{u} = 0$

(5) $\bar{u} \cdot \bar{u} = \|\bar{u}\|^2$

Ex $\bar{u} = \langle 2, -2 \rangle$ $\bar{v} = \langle 5, 8 \rangle$ $w = \langle -4, 3 \rangle$

show #2

$\bar{v} + \bar{w} = \langle 1, 11 \rangle$

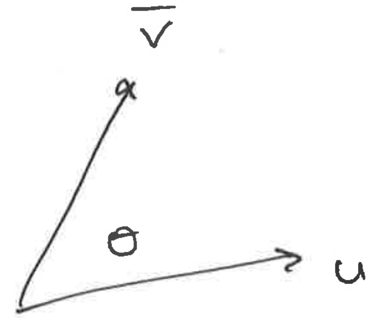
LS $= \bar{u} \cdot \bar{v} + \bar{w} = \langle 2, -2 \rangle \cdot \langle 1, 11 \rangle = 2 - 22 = -20$

$\bar{u} \cdot \bar{v} = \langle 2, -2 \rangle \cdot \langle 5, 8 \rangle = 10 - 16 = -6$

$\bar{u} \cdot \bar{w} = \langle 2, -2 \rangle \cdot \langle -4, 3 \rangle = -8 - 6 = -14$

RS. $\bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{w} = -6 - 14 = -20 =$ LS.

Alternate Def^{n}

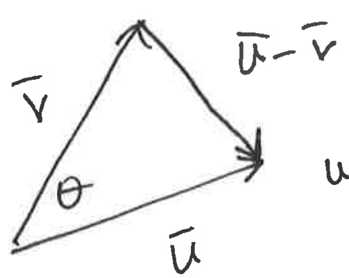


$$\bar{u}\cdot\bar{v} = \|\bar{u}\|\,\|\bar{v}\|\cos\theta$$

$\|\bar{u}\|, \|\bar{v}\|$ magnitude

θ angle between them

How this comes to be. Cosine law



$$\|\bar{u}-\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\|\bar{u}\|\,\|\bar{v}\|\cos\theta$$

use # (5) prop

$$(\bar{u}-\bar{v})\cdot(\bar{u}-\bar{v}) = \bar{u}\cdot\bar{u} + \bar{v}\cdot\bar{v} - 2\|\bar{u}\|\,\|\bar{v}\|\cos\theta$$

use prop (2)

$$\cancel{\bar{u}\cdot\bar{u}} - \underbrace{\bar{u}\cdot\bar{v} - \bar{v}\cdot\bar{u}}_{\text{prop \#1}} + \cancel{\bar{v}\cdot\bar{v}} = \cancel{\bar{u}\cdot\bar{u}} + \cancel{\bar{v}\cdot\bar{v}} - 2\|\bar{u}\|\,\|\bar{v}\|\cos\theta$$

$$-2\bar{u}\cdot\bar{v} = -2\|\bar{u}\|\,\|\bar{v}\|\cos\theta$$

$$\Rightarrow \quad \bar{u}\cdot\bar{v} = \|\bar{u}\|\,\|\bar{v}\|\cos\theta$$

one important feature of this alternate def^{n} is if $\bar{u} \perp \bar{v}$ the angle between $\bar{u}$ & $\bar{v}$ is $\pi/2$ & $\cos \pi/2 = 0$

so if $\bar{u} \perp \bar{v}$ then $\bar{u} \cdot \bar{v} = 0$

<u>ex 3</u> Are the vectors

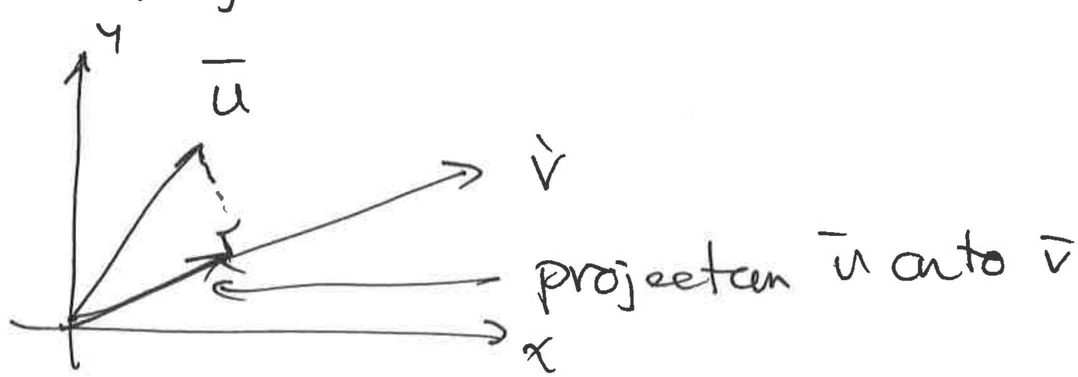
$\bar{u} = \langle 1, 2, -3 \rangle$ $\bar{v} = \langle 2, 3, 1 \rangle$ $\perp$

$\bar{u} \cdot \bar{v} = 2 + 6 - 3 = 5 \neq 0$ so no

Are $\bar{u} = \langle 1, 2, -4 \rangle$ $\bar{v} = \langle 2, 3, 2 \rangle$

$\bar{u} \cdot \bar{v} = 2 + 6 - 8 = 0$ ✓ so yes $\bar{u} \perp \bar{v}$

we will find that the dot Product will be useful in projecting one vector onto another



Cross Product

Given $\bar{u} = \langle u_1, u_2, u_3 \rangle$ $\bar{v} = \langle v_1, v_2, v_3 \rangle$

$$\bar{u} \times \bar{v} = \langle u_2 v_3 - u_3 v_2, \; u_3 v_1 - u_1 v_3, \; u_1 v_2 - u_2 v_1 \rangle$$

There is a better way to calculate this

1st Define determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Ex $$\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = (1)(5) - (1)(2) = 5 - 2 = 3$$

$$\bar{u} \times \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \bar{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \bar{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \bar{k}$$

take out row column with i — take out row, column j — similar w/ k

Ex $\quad \bar{u} = \langle 1, -2, 1 \rangle \qquad \bar{v} = \langle 3, 1, -2 \rangle$

$$\bar{u} \times \bar{v} = \begin{vmatrix} \vec{i} & \bar{j} & \bar{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} \bar{j} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \bar{k}$$

$$= (4-1)\,\vec{i} - (-2-3)\,\bar{j} + (1-(-6))\,k$$

$$= 3\vec{i} + 5\bar{j} + 7\bar{k}$$

<u>Note:</u> $\quad \bar{u} \times \bar{v} \perp \bar{u}$ & $\bar{v}$

check

$\bar{u} \times \bar{v} \cdot \bar{u} = \langle 3, 5, 7 \rangle \cdot \langle 1, -2, 1 \rangle = 3 - 10 + 7 = 0$ ✓

$\bar{u} \times \bar{v} \cdot \bar{v} = \langle 3, 5, 7 \rangle \cdot \langle 3, 1, -2 \rangle = 9 + 5 - 14 = 0$ ✓

Note $\quad \bar{u} \times \bar{u} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{vmatrix}$

$$= \underbrace{\begin{vmatrix} -2 & 1 \\ -2 & 1 \end{vmatrix}}_{=0} \bar{i} - \underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}_{=0} \bar{j} + \underbrace{\begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix}}_{=0} \bar{k} = \langle 0, 0, 0 \rangle$$

Properties Let $\bar{u}, \bar{v}, \bar{w}$ vectors c scalar

(i) $\bar{u} \times \bar{v} = -\bar{v} \times \bar{u}$

(ii) $\bar{u} \times (\bar{v} + \bar{w}) = \bar{u} \times \bar{v} + \bar{u} \times w$

(iii) $c(\bar{u} \times \bar{v}) = (c\bar{u}) \times \bar{v} = \bar{u} \times (c\bar{v})$

(iv) $\bar{u} \times \bar{0} = \bar{0} \times \bar{u} = \bar{0}$

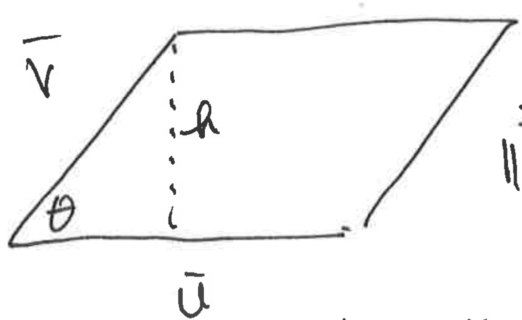
(v) $\bar{u} \times \bar{u} = \bar{0}$

vi $\bar{u} \cdot (\bar{v} \times \bar{w}) = (\bar{u} \times \bar{v}) \cdot \bar{w}$

Notes Similar to $\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos\theta$

we have

$$\|\bar{u} \times \bar{v}\| = \|\bar{u}\| \|\bar{v}\| \sin\theta$$



$\dfrac{h}{\|\bar{v}\|} = \sin\theta \Rightarrow h = \|\bar{v}\| \sin\theta$

$$A = \|\bar{u}\| h = \|\bar{u}\| \|\bar{v}\| \sin\theta$$

so $\|\bar{u} \times \bar{v}\|$ = area of parallelogram
important in Calc 3