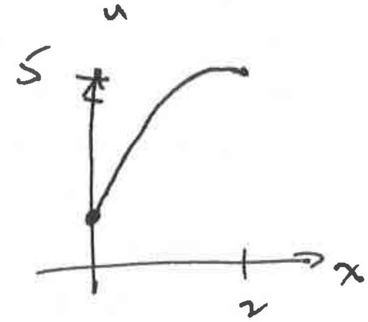
Math 4315 - PDE's   Monday Nov 7/16

Solve $\quad u_t = u_{xx} \quad 0 < x < 2$

$u(0,t) = 1 \quad u(2,t) = 5$

$u(x,0) = 4x + 1 - x^2$



Assume sep. sol^{n}'s $\quad u = T(t)\,X(x)$

Again $\quad \dfrac{T'}{T} = \dfrac{X''}{X} = \lambda$

B.C. $\quad T(t)\,X(0) = 1$

$T(t)\,X(2) = 5$

$\left.\begin{array}{l} X(0) \\ X(2) \end{array}\right\}$ these are #'s

So this can't happen

If we could "bring down" the B.C.'s we could maybe turn this problem into one we know how to solve

Try $\quad u = a + v \qquad a\ \#$

this will only bring down 1 B.C.

Next try $u = ax + b + v$

where $v(x,t)$ will be the new variable

$u(0,t) = 1 \Rightarrow a(0) + b + v(0,t) = 1$

$u(2,t) = 5 \Rightarrow a(2) + b + v(2,t) = 5$

we want $v(0,t) = v(2,t) = 0$

so $a(0) + b = 1$, $2a + b = 5$

$\Rightarrow \quad a = 2 \quad b = 1$

so $u = 2x + 1 + v(x,t)$

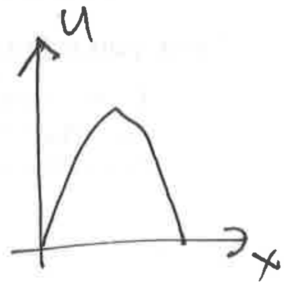
So the B.C. are good. Now what happens to the IC & PDE

I.C. $u(x,0) = 2x + 1 + v(x,0)$

we know this

$4x + 1 - x^2 = 2x + 1 + v(x,0)$

$v(x,0) = 2x - x^2$



Now the PDE

if $u = 2x + 1 + v$

$u_t = v_t, \quad u_{xx} = v_{xx}$

So PDE & IC/BCs are

$$v_t = v_{xx} \quad 0 < x < 2$$

$$v(0,t) = v(2,t) = 0$$

$$v(x,0) = 2x - x^2$$

This problem we know how to solve

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{2}\right)^2 t} \sin \frac{n\pi}{2} x$$

$$b_n = \frac{2}{2} \int_0^2 (2x - x^2) \sin \frac{n\pi}{2} x \, dx$$

& $u = 2x + 1 + \displaystyle\sum_{n=1}^{\infty} \frac{16(1 - \cos n\pi)}{n^3 \pi^3} e^{-\left(\frac{n\pi}{2}\right)^2 t} \sin \frac{n\pi}{2} x$

Suppose we have prescribed flux B.C.

Say $u_x(0,t) = 1, \quad u_x(1,t) = 0$

Sep. Sol^n $\quad u = T(t)\,X(x)$

then $\quad u_x = T(t)\,X'(x)$

so $\quad u_x(0,t) = 1 \Rightarrow T(t)\,X'(0) = 1 \quad \Leftarrow$ this one is a prob

$\quad u_x(2,t) = 0 \Rightarrow T(t)\,X'(2) = 0 \Rightarrow X'(2) = 0$

Again try

$$u = ax + b + v$$

PDE is ok.

$$u_x = a + v_x$$

and we can't choose a to bring both

$\cancel{u_x(1,t)}$ $u_x(0,t)$, $u_x(1,t)$ to zero we could try

$$u = ax^2 + bx + v$$

$$u_x = 2ax + b + v_x$$

& $u_x(0,t) = b + v_x(0,t)$

$$1 = b + 0 \quad \text{so} \quad b = 1$$

$$u_x(1,t) = 2a(1) + b + v_x(1,t)$$

$$0 = 2a + 1 + 0$$

$$\Rightarrow \quad a = -1/2$$

so $u = -\frac{1}{2}x^2 + x + v$ would give

$$v_x(0,t) = v_x(1,t) = 0$$

but PDE changes

$$u_t = v_t, \quad u_{xx} = -1 + v_{xx}$$

so $\quad u_t = u_{xx}$

becomes $\quad v_t = v_{xx} - 1 \quad$ this poses a problem

New try

$$u = at + bx^2 + cx + v$$

$u_t = a + v_t, \quad u_{xx} = 2b + v_{xx}$

$u_t = u_{xx} \Rightarrow \quad a + v_t = 2b + v_{xx} \quad$ pick $a = 2b$

so $\quad v_t = v_{xx}$

and everything will work out.