

# Spring 2026 – Math 3331

## Second Order ODEs - Reduction of Order

We now consider solving

$$a(x)y'' + b(x)y' + c(x)y = 0. \quad (1)$$

For example, the reader can verify that

$$y'' - 3y' + 2y = 0 \quad (2)$$

has as independent solutions  $y_1 = e^x$  and  $y_2 = e^{2x}$ . Suppose that we are given just one of these, say  $y = e^x$ , could we find the other. The answer is yes! One way is a method called *reduction of order*. Simple put, we let  $y_2 = e^x u$  for some  $u = u(x)$ . We illustrate with several examples.

*Example 1.*

Find the second independent solution to

$$y'' - 3y' + 2y = 0 \quad (3)$$

if one solution is  $y_1 = e^x$ . Here we let

$$y = e^x u \quad (4)$$

so

$$\begin{aligned} y' &= e^x u' + e^x u, \\ y'' &= e^x u'' + 2e^x u' + e^x u \end{aligned} \quad (5)$$

Substituting into (3) gives

$$\begin{aligned} y'' - 3y' + 2y &= 0 \\ e^x(u'' + 2u' + u) - 3e^x(u' + u) + 2e^x u &= 0 \\ u'' + 2u' + u - 3u' - 3u + 2u &= 0 \\ u'' - u' &= 0. \end{aligned} \quad (6)$$

Note that this has no  $u$ . If we let  $u' = v$  then  $u'' = v'$  and we have

$$v' - v = 0 \quad (7)$$

and first order separable ODE. We integrate giving

$$v = c_1 e^x \quad (8)$$

Now  $v = u'$  so

$$u' = c_1 e^x \Rightarrow u = c_1 e^x + c_2 \quad (9)$$

Since  $y = e^x u$  then

$$y = e^x u = e^x (c_1 e^x + c_2) \quad (10)$$

and expanding gives

$$y = c_1 e^{2x} + c_2 e^x \quad (11)$$

Here we have both independent solutions. The one we wanted was  $y = e^{2x}$ . To simplify our calculation (and clutter) we set the first and second constants of integration to either 0 or 1. In this case  $c_1 = 1$  and  $c_2 = 0$ .

*Example 2.*

Find the second independent solution to

$$y'' - 2y' + y = 0 \quad (12)$$

if one solution is  $y_1 = e^x$ . This is the same as the previous example except the ODE has changed. Here we let

$$y = e^x u \quad (13)$$

so

$$\begin{aligned} y' &= e^x u' + e^x u, \\ y'' &= e^x u'' + 2e^x u' + e^x u \end{aligned} \quad (14)$$

Substituting into (12) gives

$$\begin{aligned} y'' - 2y' + y &= 0 \\ e^x(u'' + 2u' + u) - 2e^x(u' + u) + e^x u &= 0 \\ u'' + 2u' + u - 2u' - 2u + u &= 0 \\ u'' &= 0. \end{aligned} \quad (15)$$

We could let  $u' = v$  giving a first order ODE but this ODE  $u''$  is easily integrated (twice) giving

$$u = c_1 x + c_2 \quad (16)$$

and we set  $c_1 = 1$  and  $c_2 = 0$ . Thus,

$$y_2 = e^x u = x e^x \quad (17)$$

and the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^x + c_2 x e^x \end{aligned} \quad (18)$$

*Example 3.*

Find the second independent solution to

$$x^2 y'' - 3xy' + 3y = 0 \quad (19)$$

if one solution is  $y_1 = x$ . Here we let

$$y = xu \quad (20)$$

so

$$\begin{aligned} y' &= xu' + u, \\ y'' &= xu'' + 2u' \end{aligned} \quad (21)$$

Substituting into (3) gives

$$\begin{aligned} x^2 y'' - 3xy' + 3y &= 0 \\ x^2(xu'' + 2u') - 3x(xu' + u) + 3xu &= 0 \\ x^3 u'' + 2x^2 u' - 3x^2 u' - 3xu + 3xu &= 0 \\ x^3 u'' - x^2 u' &= 0 \\ xu'' - u' &= 0. \end{aligned} \quad (22)$$

Note that this has no  $u$ . If we let  $u' = v$  then  $u'' = v'$  and we have

$$xv' - v = 0 \quad (23)$$

and first order separable ODE. We integrate giving

$$v = x \quad (24)$$

Now  $v = u'$  so

$$u' = x \Rightarrow u = \frac{1}{2}x^2 \quad (25)$$

Since  $y_2 = xu$  then

$$y_2 = xu = x \left( \frac{1}{2}x^2 \right) = \frac{1}{2}x^3 \quad (26)$$

and expanding gives

$$y = c_1x + c_2x^3 \quad (27)$$

noting the factor of  $1/2$  has been absorbed into the  $c_2$ .