International Journal of Modern Science and Technology, 2020;5(2):40-44.
***ISSN: 2456-0235*.**
www.ijmst.co

Research Article

Properties of Norm-attainable Classes and Applications to Signal Processing

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Abstract

Properties of elementary operators have been studied over a long period of time. Such properties include norms, spectrum, positivity, numerical ranges among others. Characterization of elementary operators has been given considerations in different classes. In the present work authors give norm properties of elementary operators in norm-attainable classes. Moreover, we give applications to signal processing.

Keywords: Elementary operators; Norm-attainable classes; Spectrum; Signal Processing.

Introduction

Studies on properties of elementary operators [1-4] has been of interest to many researchers. Such properties include norms [5], spectrum [6], positivity [7], numerical [8] ranges among others. Characterization of elementary operators has been given considerations in different classes [9]. Let $\mathcal{B}(E)$ be the norm-attainable class of all bounded linear operators on a Hilbert space *E*. For *n*-tuples $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots B_n)$ of operators on *E*, let $R_{A,B}$ denote the operator on $\mathcal{B}(E)$ defined by $R_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i$.

For $A, B \in \mathcal{B}(E)$, authors put $U_{A,B} = R_{(A,B),(B,A)}$. We seek in the present work, authors prove that
$$co\left\{\sum_{i=1}^{n}\alpha_i\beta_i : (\alpha_1, \dots, \alpha_n) \in V(A), (\beta_1, \dots, \beta_n) \in V(B)\right\}^{-} \subset W_0(R_{A,B}|J)$$
where $V(\cdot)$ is the joint spatial numerical range, $W_0(\cdot)$ is the algebraic numerical range and *J* is a norm ideal of $\mathcal{B}(E)$.

Also, authors show that $w(U_{A,B}|J) \geq 2(\sqrt{2}-1)w(A)w(B)$, for $A, B \in \mathcal{B}(E)$ and *J* is a norm ideal of $\mathcal{B}(E)$, where $w(\cdot)$ is the numerical radius. Now, if *E* is a Hilbert space, we show that the lower bound estimate $\|U_{A,B}|J\| \geq 2(\sqrt{2}-1)\|A\|\|B\|$ holds, if: *J* is a standard operator algebra of $\mathcal{B}(E)$ and *A*,*B*∈ *J* and also if *J* is a norm ideal of $\mathcal{B}(E)$ and $A, B \in \mathcal{B}(E)$.

Research methodology

In this section, we give the methods and techniques used to generate our results. All operators considered here are linear bounded operators on a Hilbert space *E*. We adopt the following notations in this work. If $M \subset \mathbb{C}$, we denote by M^{-}, co *M* and $\overline{M}$, respectively the closure of *M*, the convex hull of *M*, and the set $\{\overline{\lambda} : \lambda \in M\}$. For $(x,f) \in E \times E^*$, we denote by $x \otimes f$ the operator on *E* given by $(x \otimes f)(y) = f(y)x$. If *E* is a Hilbert space and if $x, y \in E$, we denote by $x \otimes y$ the operator on *E* given by $(x \otimes y)(z) = \langle z, y\rangle x$. If $K, L \subset \mathbb{C}^n$, we put
$$K \circ L = \sum_{i=1}^{n}\alpha_i\beta_i - 3 : (\alpha_1, \dots, \alpha_n) \in K, (\beta_1, \dots$$

Definition 2.1

Let Ω be a complex unital Banach algebra with identity *I* and let *A*∈ Ω.

We define: The spectrum of *A* by:
$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible in } \Omega\}$$

The spectral radius of *A* by:
$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

Received: 12.01.2020; Received after Revision: xx.xx.2020; Accepted: xx.xx.2020; Published: 20.02.2020

The set of states on Ω by: $\mathcal{P}(\Omega) = \{ f \in \Omega^*: f(I) = \| f \| = 1 \}$

The algebraic numerical range of *A* by: $W_0(A) = \{ f(A): f \in \mathcal{P}(\Omega) \}$

The numerical radius of *A* by: $w(A) = \sup\{|\lambda| : \lambda \in W_0(A)\}$

A is called convexiod if $W_0(A)$=co σ (*A*).

It is known in [10] that $W_0(A)$ is convex and compact (this result follows at once from the corresponding properties of the set of states) and contains σ (*A*). If $\Omega = \mathcal{B}(E)$ and *E* is a Hilbert space, then *w*(*A*)=\| *A*| iff *r*(*A*)=| *A*|.

Definition 2.2

For $A \in \mathcal{B}(E)$, define the spatial numerical range of *A* by: $V(A) = \{ f(Ax): (x, f) \in \Pi \}$ where $\Pi = \{(x, f) \in E \times E^*: \|x\| = \| f\| = f(x) = 1\}$.

Definition 2.3

For *n*-tuples $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ of operators on *E*, we define: the joint spatial numerical range of *A* (see [4]) by: $V(A) = \{(f(A_1x), \ldots, f(A_nx)): (x, f) \in \Pi\}$, the joint numerical range of *A* by: $W(A) = \{(\langle A_1x, x\rangle, \ldots, \langle A_nx, x\rangle): x \in E, \|x\| = 1\}$,

Definition 2.4

For $A, B \in \mathcal{B}(E)$, define the particular elementary operators: the left multiplication operator by: $\forall X \in \mathcal{B}(E): L_A(X) = AX$, the right multiplication operator by: $\forall X \in \mathcal{B}(E): R_B(X) = XB$, the generalized derivation (induced by *A*,*B*) by $\delta_{A,B} = L_A - R_B$, the elementary multiplication operator (induced by *A*,*B*) by $M_{A,B} = L_A R_B$, the operator $U_{A,B}$, by $U_{A,B} = M_{A,B} + M_{B,A}$.

Remark 2.5

Without loss of generality, $T_{A,B}$ will stand for any one of the above linear operators. Let *J* be a standard operator algebra or a norm ideal of $\mathcal{B}(E)$. Note that a standard operator algebra of $\mathcal{B}(E)$ is a subalgebra of $\mathcal{B}(E)$ associated with the usual operator norm and containing all finite rank operators, and a norm ideal of $\mathcal{B}(E)$ is a two-sided ideal of $\mathcal{B}(E)$ associated with a symmetric norm ideal (which satisfies axioms like those in Hilbert space case. We denote by $\|\cdot\|_J$ the norm on *J*. If *J* is a norm ideal, then $T_{A,B}(J) \subset J$, so we can define the operator $T_{J,A,B}$ on *J* by $T_{J,A,B}(X) = T_{A,B}(X)$. If *J* is a standard operator algebra and *A*,*B*∈ *J*, define $U_{J,A,B}: J \to J$ by $U_{J,A,B}(X) = U_{A,B}(X)$.

For any $A, B \in \mathcal{B}(E)$, we have proved that: and $co(W(A) \circ W(B))^- \subset W_0(R_{J,A,B})$, $W_0(\delta_{J,A,B}) = W_0(A) - W_0(B)$.

Results and discussion

In this section, we give the results of our study. We assume that *J* is a norm ideal.

Proposition 3.1

Consider *E* as a Hilbert space and let *A* and *B* be two *n*-tuples of operators on *E*. Then $co(W(A) \circ W(B))^- \subset W_0(R_{J,A,B})$.

Proof

For $J = \mathcal{B}(E)$ (resp. $J = \mathcal{C}_p(E)$, the Schatten *p*-ideal), then the result is obtained in [11]. For any norm ideal *J*, the proof is analogous to that of [5].

Proposition 3.2

Let *A* and *B* be two *n*-tuples of operators on *E*. Then $co(V(A) \circ V(B))^- \subset W_0(R_{J,A,B})$.

Proof

Let (*x*,*f*),(*y*,*g*)∈ Π . Define the linear functional *h* on $\mathcal{B}(J)$ by: $h(F) = f(F(x \otimes g)y), \quad F \in \mathcal{B}(J)$

We have *h*(*I*)=*f*(*x*)*g*(*y*)=1, and since $\|x \otimes g\|_J = \|x \otimes g\| = \|x\|\|g\| = 1$, then:

$$\begin{cases} |h(F)| \le \|F(x \otimes g)y\| \\ \quad \le \|F(x \otimes g)\| \\ \quad \le \|F(x \otimes g)\|_J \\ \quad \le \|F\| \|x \otimes g\|_J \\ \quad \le \|F\|. \end{cases}$$

So *h*(*I*)=| *h*| =1; thus *h* is a state on $\mathcal{B}(J)$. It is obvious that $h(R_{J,A,B}) = \sum_{i=1}^n f(A_i x) g(B_i y)$, therefore $V(A) \circ V(B) \subset W_0(R_{J,A,B})$. Since $W_0(R_{J,A,B})$ is closed and convex, the result follows easily.

Lemma 3.3

Let $A \in \mathcal{B}(E)$. Then $W_0(L_{J,A}) = W_0(R_{J,A}) = W_0(A)$.

Proof

The inclusion $co\, V(A)^- \subset W_0(L_{J,A})$ follows immediately from Proposition 3.2. Then $W_0(A) = co\, V(A)^- \subset W_0(L_{J,A})$. Now, let *f* be a state on $\mathcal{B}(J)$. Define the linear functional *g* on $\mathcal{B}(E)$ by *g*(*X*)= *f*($L_{J,X}$). By a simple computation, we find that *g* is bounded a stateon $\mathcal{B}(E)$, sothat $g(A) = f(L_{J,A}) \in W_0(A)$. Thus $W_0(L_{J,A}) \subset W_0(A)$, therefore $W_0(L_{J,A}) = W_0(A)$. By the same argument, we find also $W_0(R_{J,A}) = W_0(A)$.

Theorem 3.4

Let $A, B \in \mathcal{B}(E)$. Then $W_0(\delta_{J,A,B}) = W_0(A) - W_0(B)$.

Proof

By Theorem 2, we have $co(V(A) - V(B))^- \subset W_0(\delta_{J,A,B})$. Then

$$W_0(A) - W_0(B) = co\, V(A)^- - co\, V(B)^- = co(V(A) - V(B))^- \subset W_0(\delta_{J,A,B})$$

$$W_0(\delta_{J,A,B}) = W_0(L_{J,A} - R_{J,B}) \subset W_0(L_{J,A}) - W_0(R_{J,B}) = W_0(A) - W_0(B)$$

Corollary 3.5

Let $A, B \in \mathcal{B}(E)$. Then $W_0(A)W_0(B) \subset W_0(M_{J,A,B})$, and thus $w(M_{J,A,B}) \geq w(A)w(B)$.

Proof

By Proposition 3.2, we obtain $co(V(A)V(B))^- \subset W(M_{J,A,B})$.

Then we have:

$$W_0(A)W_0(B) = co\, V(A)^- co\, V(B)^- = (co\, V(A) co\, V(B))^- \subset co(V(A)V(B))^- \subset W_0(M_{J,A,B})$$

The inequality follows immediately from this inclusion.

Proposition 3.6

Let $A, B \in \mathcal{B}(E)$. Then $w(U_{J,A,B}) \geq 2(\sqrt{2} - 1)w(A)w(B)$.

Proof

We may assume, without loss of the generality, that *w*(*A*)=*w*(*B*)=1.

For any (*x*,*f*),(*y*,*g*) in Π, we have $f(Ax)g(By) + f(Bx)g(Ay) \in V(A,B) \circ V(B,A)$Since $V(A,B) \circ V(B,A) \subset W_0(U_{J,A,B})$, then $w(U_{J,A,B}) \geq |f(Ax)g(By) + f(Bx)g(Ay)|$

Applying inequality (1) for (*y*,*g*)=(*x*,*f*), we obtain:

$$w(U_{J,A,B}) \geq 2|f(Ax)||f(Bx)| \quad (2)$$

Let (x_n,f_n) and (y_n,g_n) be two sequences in Πsuch that: $\lim|f_n(Ax_n)| = w(A) = 1 = w(B) = \lim|g_n(By_n)|$For (*x*,*f*)=($x_n$,$f_n$) and (*y*,*g*)=($y_n$,$g_n$), inequality (1) yields:

$$w(U_{J,A,B}) \geq |f_n(Ax_n)g_n(By_n) + f_n(Bx_n)g_n(Ay_n)|$$

Thus,

$$w(U_{J,A,B}) \geq |f_n(Ax_n)g_n(By_n)| - |f_n(Bx_n)g_n(Ay_n)|$$

Applying inequality (2) twice for (*x*,*f*)=(x_n,f_n) and for (*y*,*g*)=(y_n,g_n) we obtain:

$$\begin{cases} w(U_{J,A,B}) \geq 2|f_n(Ax_n)||f_n(Bx_n)|. \\ w(U_{J,A,B}) \geq 2|g_n(Ay_n)||g_n(By_n)| \end{cases}$$

Since the two complex sequences (f_n(Bx_n)) and (g_n(Ay_n)) are bounded, we can extract a convergent subsequence from each one. We can put $\alpha = \lim|f_n(Bx_n)|$ and $\beta = \lim|g_n(Ay_n)|$.

Letting *n*→+∞ in (4), (5) and (6), we obtain,$w(U_{J,A,B}) \geq \max\{1 - |\alpha\beta|, 2|\alpha|, 2|\beta|\}$Therefore,

$$\begin{cases} w(U_{J,A,B})^2 + 4w(U_{J,A,B}) \geq 4|\alpha\beta| + 4(1 - |\alpha\beta|) \\ \geq 4. \end{cases}$$

Thus we have $w(U_{J,A,B}) \geq 2(\sqrt{2} - 1)$.

In this point, we assume that *E* is a Hilbert space. Let $A, B \in \mathcal{B}(E)$. We assume that if *J* is a standard operator algebra, then *A*,*B*∈ *J*.

Remark 3.7

We define the numerical range of *A***B* relative to *B* by:

$$W_B(A^*B) = \{\lambda \in \mathbb{C}: \lambda = \lim\langle A^*Bx_n, x_n\rangle,\ \lim\|Bx_n\| = \|B\|,\ \|x_n\| = 1\}$$

This concept of this numerical range is introduced by the most interesting properties of W_B (*A***B*) are given as below.

$W_B(A^*B)$ is not empty and compact subset of $\mathbb{C}$,

the relation $\inf_{\lambda\in\mathbb{C}} \|B - \lambda A\| = \|B\|$ holds iff $0\in W_B(A^*B)$.

Theorem 3.8

We have that: $\|U_{J,A,B}\| \geq \sup\{|\langle Ax,y\rangle\langle Bu,v\rangle + \langle Bx,y\rangle\langle Au,v\rangle| - 3 : \|x\| = \|y\| = \|u\| = \|v\| = 1\}$ and $\|U_{J,A,B}\| \geq 2w(A^*B)$.

Proof

Since $\|x\otimes v\|_J = \|x\otimes v\| = \|x\|\,\|v\| = 1$, and since $\|X\|_J \geq \|X\|$, for any $X\in J$,

$$\begin{aligned}\|U_{J,A,B}\| &\geq \|A(x\otimes v)B + B(x\otimes v)A\|_J\\ &\geq \|Ax\otimes B^*v + Bx\otimes A^*v\|\\ &\geq \|\langle Bu,v\rangle Ax + \langle Au,v\rangle Bx\|\\ &\geq |\langle Ax,y\rangle\langle Bu,v\rangle + \langle Bx,y\rangle\langle Au,v\rangle|\end{aligned}$$

Let x be a unit vector in E such that $Ax\neq 0$. Using (i), we obtain, $\|U_{J,A,B}\| \geq |(1/\|Ax\|)\langle A^*Bx,x\rangle\|Ax\| + \langle A^*Bx,x\rangle\|Ax\|$ then we can deduce immediately that $\|U_{J,A,B}\| \geq 2|\langle A^*Bx,x\rangle|$, for any unit vector x in E. So $\|U_{J,A,B}\| \geq 2w(A^*B)$.

Corollary 3.9

We have the following property: $\|U_{J,A,B}\| \geq 2(\sqrt{2}-1)\|A\|\|B\|$.

Proof

We may assume, without loss of the generality, that $|A|=|B|=1$.
Let $\lambda\in W_B(A^*B)$ and $\mu\in W_A(B^*A)$. Then, there exist two sequences (x_n) and (y_n) of unit vectors in E such that $\lim\|Bx_n\| = \lim\|Ay_n\| = 1$, and $\lim\langle A^*Bx_n,x_n\rangle = \lambda$, $\lim\langle B^*Ay_n,y_n\rangle = \mu$.
By Lemma 3.3.(i), we have:

$$\|U_{J,A,B}\| \geq \left|\frac{1}{\|Ay_n\|\|Bx_n\|}\langle A^*By_n,y_n\rangle\langle B^*Ax_n,x_n\rangle + \|Ay_n\|\|Bx_n\|\right|$$

Letting $n\to+\infty$, we get $\|U_{J,A,B}\| \geq |1+\overline{\lambda}\overline{\mu}\| = |1+\lambda\mu|$.

On the other hand, by Lemma 1.(ii), we have $\|U_{J,A,B}\| \geq \max\{2|\lambda|, 2|\mu|\}$; therefore

$\|U_{J,A,B}\| \geq \max\{|1+\lambda\mu|, 2|\lambda|, 2|\mu|\}$,

and by the same argument as in the proof of Theorem 3, we obtain the inequality.

We note that the above Corollary 3.9 is proved [10] in the particular case where J is standard operator algebra, but here, we have obtained it, in a more general situation by a direct proof.

Theorem 3.10

If A and B are not zero, we have: $\|U_{J,A,B}\| \geq \sup\left\{\left|\|A\|\|B\| + \frac{\lambda\mu}{\|A\|\|B\|}\right|, \lambda\in W_B(A^*B), \mu\in W_A(B^*A)\right\}$

Proof

Let $\lambda\in W_B(A^*B)$ and $\mu\in W_A(B^*A)$. By the same argument as in the proof of the Theorem 3.3, we obtain $\|U_{J,A,B}\| \geq |\|A\|\|B\| + (\lambda\mu/\|A\|\|B\|)|$.

Applications to signal processing

We consider the shifts T and T^* as operator generators of filters for signal and image processing on meshes with semigroup structures. We know that some standard operations in digital signal processing have suitable analogs among those filters. That is, we have representations of operations as operators written by means of the shifts. In practice, we have used these filters for denoising signals and in combination with matricization and tensorization operations of the Fourier transformation and a wavelet expansion for further signal and image processing.

Conclusions

In the present work, authors have studied properties of elementary operators which have also been considered over a long period of time. Such properties include norms, spectrum, positivity, numerical ranges among others. Characterization of elementary operators has been given considerations in different classes. In this paper we have given norm properties of elementary operators in norm-attainable classes. Moreover, authors have given applications to signal processing.

Conflict of interest

The authors declare no conflict of interests.

References

[1] Barraa M, Boumazgour M. Inner derivation and norm equality. Proc Amer Math Soc 2020;130(7):471-6.

[2] Halmos PR. A Hilbert Space Problem Book,Springer Verlag New York; 1970.

[3] Lumer G. Semi-inner product spaces. Trans Amer Math Soc 2016;100(5):29-43.

[4] McIntosh A. Heinz inequalities and perturbation of spectral families. Macquarie Mathematics Reports 2006;291(2):79-86.

[5] Taylor J. A joint spectrum for several commuting operators. J Funct Anal 2019;6(2):172-91.

[6] Okelo NB. On Characterization of Various Finite Subgroups of Abelian Groups. International Journal of Modern Computation, Information and Communication Technology 2018;1(5):93-8.

[7] Okelo NB. On Normal Intersection Conjugacy Functions in Finite Groups. International Journal of Modern Computation, Information and Communication Technology 2018;1(6):111-5.

[8] Okwany I, Odongo D, and Okelo NB. Characterizations of Finite Semigroups of Multiple Operators. International Journal of Modern Computation, Information and Communication Technology 2018;1(6):116-20.

[9] Ramesh R, Mariappan R. Generalized open sets in Hereditary Generalized Topological Spaces. J Math Comput Sci 2015;5(2):149-59.

[10] Wanjara AO. On the Baire's Category Theorem as an Important Tool in General Topology and Functional Analysis. International Journal of Modern Computation, Information and Communication Technology 2019;2(4):27-31.

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