Calculus 3 - Double Integrals - Polar

Example. Find the volume under the paraboloid $z = 2 - x^2 - y^2$ and inside the cylinder $x^2 + y^2 = 1$, for $z \geq 0$.

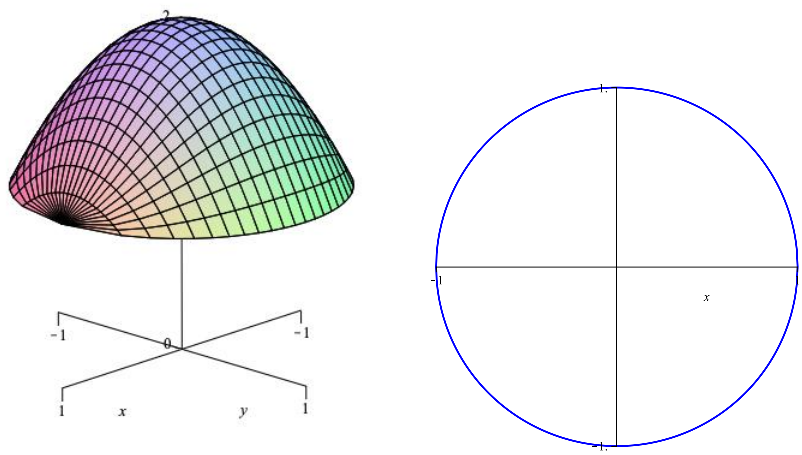


Figure 1: Region of integration

In this case, the height of the volume is given by the paraboloid $z = 2 - x^2 - y^2$ (this is the integrand) and so

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2 - x^2 - y^2) dydx \tag{1}$$

or

$$V = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (2 - x^2 - y^2) dxdy \tag{2}$$

The first integral (1) is when the rectangles are vertically and we have bottom curve to top curve whereas, in the second integral (2), we have horizontal rectangles and we have left curve to right curve.

Due to symmetry

$$\begin{aligned}
V &= 4\int_0^1 \int_0^{\sqrt{1-x^2}} (2-x^2-y^2)dydx \\
&= 4\int_0^1 2y - x^2y - \frac{2}{3}y^3\Big|_0^{\sqrt{1-x^2}} dx \\
&= 4\int_0^1 2\sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{2}{3}\left(\sqrt{1-x^2}\right)^3 dx \quad (trig\ sub\ x = \sin\theta) \\
&= \frac{3\pi}{2}
\end{aligned}$$

The integration in this problem was a little difficult and needed a trig sub. Maybe there's a better way!

Polar Coordinates

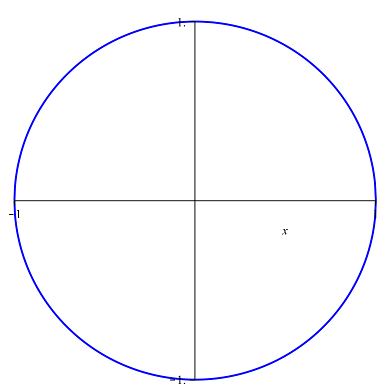
In calculus 2 we introduced polar coordinates where

$$x = r\cos\theta, \quad y = r\sin\theta \tag{3}$$

and

$$x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x} \tag{4}$$

Consider the region in the previous problem - a circle of radius 1. Let's talk about sweeping out the region



We see that

$$r = 0 \to 1, \quad \theta = 0 \to 2\pi \tag{5}$$

so what about the integral from the previous example

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2 - x^2 - y^2) dy dx. \tag{6}$$

How would it change if we used r and θ instead of x and y? Would it becomes easier?

Double Integrals in Polar Coordinates

We consider the double integral

$$V = \iint\limits_R f(x,y) dA. \tag{7}$$

This integral has three main parts:

1. the integrand
2. dA
3. limits

1. The integrand

For this part, we simplify substitute in

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{8}$$

into $f(x,y)$ and simplify. So, in general,

$$V = \iint\limits_R f(x,y) dA = \iint\limits_R f(r\cos\theta, r\sin\theta) dA. \tag{9}$$

So if the integral was say

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\, dydx. \tag{10}$$

then

$$V = \iint_R \sqrt{1 - r^2}\, dA \tag{11}$$

2. dA

In cartesian coordinates, this is $dA = dxdy$. What about in terms of dr and $d\theta$? Let us consider where the $dxdy$ came from. It came from a small area element

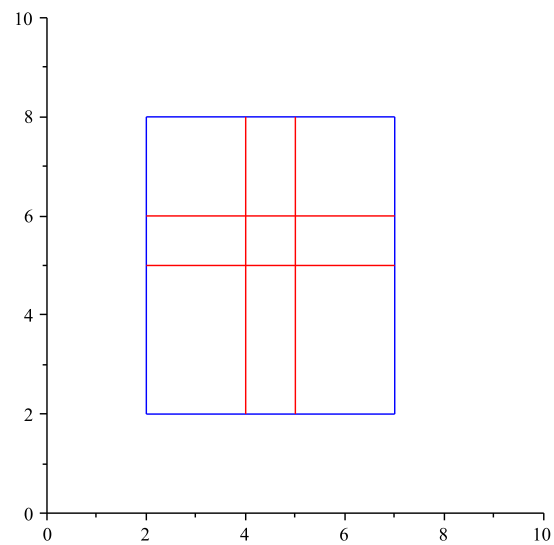


Figure 2: Cartesian region of integration

Now from our arc length formula where $s = r\theta$ then $ds = r\,d\theta$. The change is r is dr and we have

$$dA = rd\theta \times dr \tag{12}$$

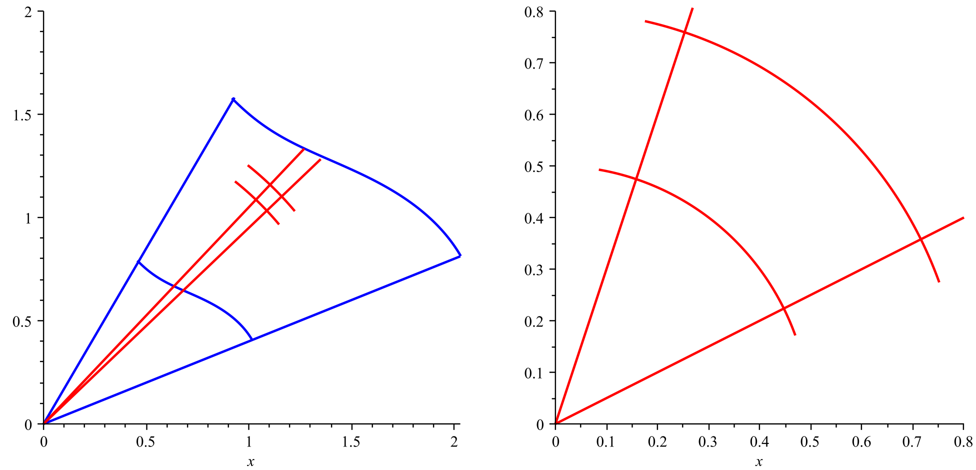


Figure 3: Polar region of integration

3. Limits of Integration

These ultimately come from the picture of the region itself.

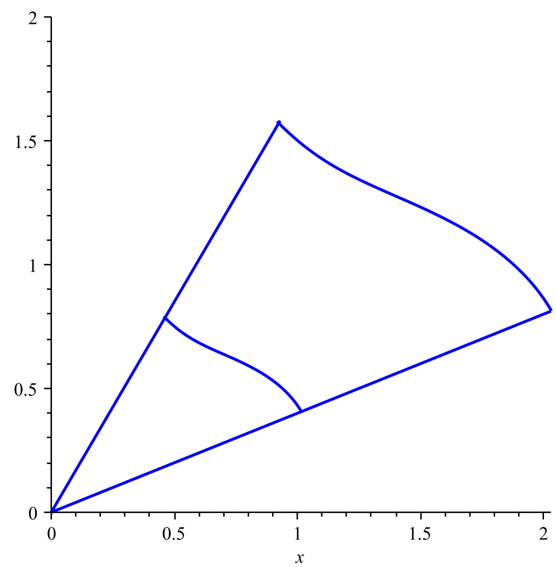


Figure 4: Cartesian region of integration

so

$$\int_{\alpha}^{\beta} \int_{r_i(\theta)}^{r_o(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta \tag{13}$$

where $r = r_i(\theta)$ is the inner curve and $r = r_o(\theta)$ is the outer curve.

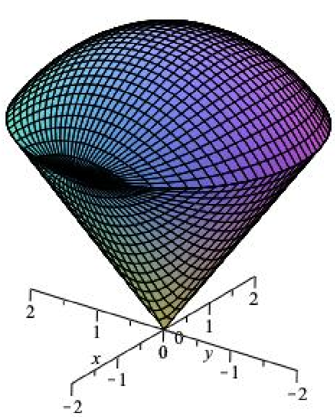
Example 1. Find the volume under the paraboloid $z = 2 - x^2 - y^2$ and inside the cylinder $x^2 + y^2 = 1$, for $z \geq 0$

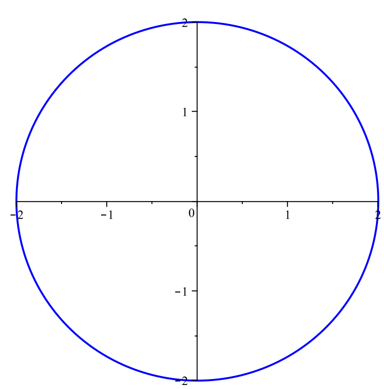
Soln. The surface is given by $z = 2 - r^2$ (this is the integrand). We are integrating over a circle of radius 1 so the volume we seek is given by

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 r - r^3 dr d\theta \\
&= \int_0^{2\pi} \frac{1}{2} r^2 - \frac{1}{4} r^4 \Big|_0^1 d\theta \\
&= \frac{1}{4} \int_0^{2\pi} d\theta \\
&= \frac{1}{4} \theta \Big|_0^{2\pi} = \frac{\pi}{2}
\end{aligned}
\tag{14}
$$

Example 2. Find the volume between the cone $z = \sqrt{x^2 + y^2}$ and the half sphere $z = \sqrt{8 - x^2 - y^2}$.

Soln. The surfaces are





The intersection of the two surface will give the region of integration so

$$\sqrt{8-x^2-y^2} = \sqrt{x^2+y^2} \quad \Rightarrow \quad 8-x^2-y^2 = x^2+y^2 \tag{15}$$

or

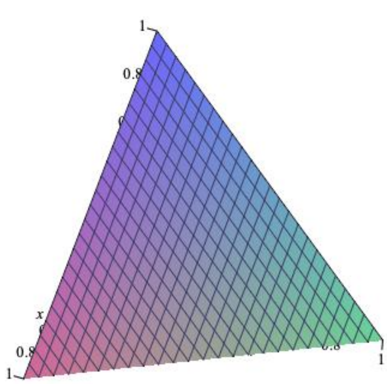
$$x^2+y^2=4 \tag{16}$$

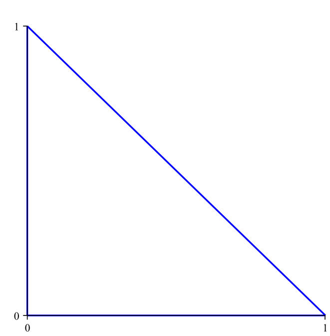
As there are two surfaces, there will be two parts in the integrand. The setup is as follows:

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^2 \left(\sqrt{8-r^2}-r\right) r dr d\theta \\ &= \int_0^{2\pi} -\frac{1}{3}\left(8-r^2\right)^{2/3} - \frac{1}{3}r^3\Big|_0^2 d\theta \\ &= \frac{16}{3}(\sqrt{2}-1)\int_0^{2\pi} d\theta \\ &= \frac{16}{3}(\sqrt{2}-1)2\pi \end{aligned} \tag{17}$$

Example 3. Find the volume of the tetrahedran bound by the planes $x+y+z=1$ $x=0$, $y=0$ and $z=0$.

Soln. The surface and region of integration is





The setup for this problem is

$$\int_0^{\pi/4} \int_0^{\frac{1}{\cos\theta+\sin\theta}} (1 - r\cos\theta - r\sin\theta)\, r dr d\theta \tag{18}$$

so polar is not the way to go!

Example 4. Pg. 995, #18 Evaluate

$$\int_0^2 \int_0^{\sqrt{4-x^2}} x dy dx \tag{19}$$

Soln.

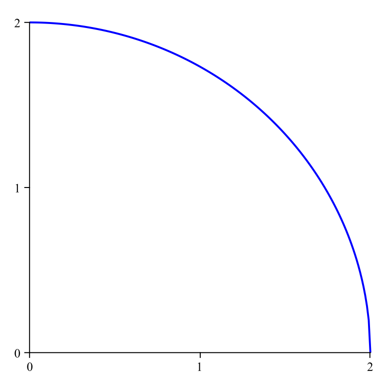
We first draw the region. We integrate with respect to y first so we go from a bottom curve to a top curve and in this case

$$y = 0 \quad \Rightarrow \quad y = \sqrt{4 - x^2} \tag{20}$$

Next we integrate with respect to x and so this is a left point and right point

$$x = 0 \quad \Rightarrow \quad x = 2 \tag{21}$$

A picture of the region is below.



Now the setup

$$\begin{aligned}\int_0^{\pi/2}\int_0^2 r\cos\theta\ rdrd\theta &= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\ drd\theta \\ &= \int_0^{\pi/2}\int_0^2 \frac{1}{3}r^3\Big|_0^2\cos\theta\ d\theta \\ &= \frac{8}{3}\sin\theta\Big|_0^{\pi/2} = \frac{8}{3}.\end{aligned} \tag{22}$$

Example 4. Pg. 995, #24 Evaluate

$$\int_0^4\int_0^{\sqrt{4y-y^2}} x^2 dxdy \tag{23}$$

Soln.

We first draw the region. We integrate with respect to x first so we go from a left curve to a right curve and in this case

$$x = 0 \quad \Rightarrow \quad x = \sqrt{4y-y^2}. \tag{24}$$

Next we integrate with respect to y and so this is a bottom point and top point

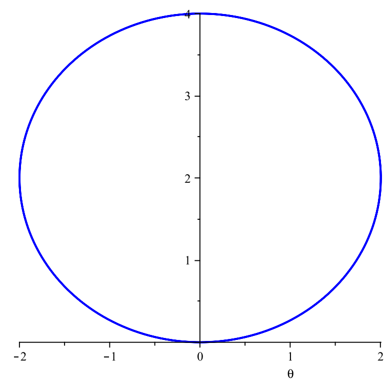
$$y = 0 \quad \Rightarrow \quad y = 4. \tag{25}$$

To get an idea of what the right curve is

$$x = \sqrt{4y-y^2} \quad \Rightarrow \quad x^2+y^2-4y = 0, \tag{26}$$

so

$$r^2 - 4r\sin\theta = 0 \quad \Rightarrow \quad r = 4\sin\theta. \tag{27}$$



Now the setup

$$
\begin{aligned}
\int_0^{\pi/2}\int_0^{4\sin\theta}(r\cos\theta)^2\ rdrd\theta &= \int_0^{\pi/2}\int_0^{4\sin\theta} r^3\cos^2\theta\ rdrd\theta \\
&= \int_0^{\pi/2}\frac{1}{4}r^4\Big|_0^{4\sin\theta}\cos^2\theta\ d\theta \\
&= 4^3\int_0^{\pi/2}\sin^4\theta\cos^2\theta\ d\theta \\
&= 4^3\cdot\frac{\pi}{32} = 2\pi.
\end{aligned} \tag{28}
$$