Systems of ODEs

A linear system of equations

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy, \tag{1}$$

can be can be written as a matrix ODE

$$\frac{d\bar{x}}{dt} = A\bar{x} \tag{2}$$

where $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\bar{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we consider solutions of the form

$$\bar{x} = \bar{c}e^{\lambda t},$$

then after substitution into (2) we obtain

$$\lambda \bar{c}\, e^{\lambda t} = A\, \bar{c}\, e^{\lambda t}$$

from which we deduce

$$(A - \lambda I)\, \bar{c} = 0. \tag{3}$$

In order to have nontrivial solutions $\bar{c}$, we require that

$$|A - \lambda I| = 0. \tag{4}$$

This is the eigenvalue-eigenvector problem. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then (4) becomes

$$\lambda^2 - (a + d)\lambda + a\,d - b\,c = 0,$$

from which we have three cases:

(i) two distinct eigenvalues

(ii) two repeated eigenvalues,

(iii) two complex eigenvalues.

Real Distinct Eigenvalues

Suppose we have eigenvalues $\lambda = \lambda_1, \lambda_2$ and associated eigenvectors $\bar{u} = \bar{u}_1, \bar{u}_2$. Then the general solution is

$$\bar{x} = c_1 \bar{u}_1 e^{\lambda_1 t} + c_2 \bar{u}_2 e^{\lambda_2 t}. \tag{5}$$

Real Repeated Eigenvalues

In this case we only have one eigenvalue and one eigenvector and thus one solution

$$\bar{x}_1 = \bar{u}_1 e^{\lambda t}. \tag{6}$$

For the second solution we use the reduction of order technique and assume the second solution is

$$\bar{x}_2 = \bar{u}\, t\, e^{\lambda t} + \bar{v}\, e^{\lambda t}. \tag{7}$$

Substitution into

$$\frac{d\bar{x}}{dt} = A\bar{x} \tag{8}$$

gives

$$\bar{u} e^{\lambda t} + \bar{u} t \lambda e^{\lambda t} + \bar{v} \lambda e^{\lambda t} = A \left(\bar{u} t e^{\lambda t} + \bar{v} e^{\lambda t} \right). \tag{9}$$

Expanding and comparing coefficients gives

$$\lambda \bar{u} = A\bar{u}, \quad \bar{u} + \lambda \bar{v} = A\bar{v}\bar{u} \tag{10}$$

or

$$(A - \lambda I)\, \bar{u} = 0, \quad (A - \lambda I)\, \bar{v} = \bar{u} \tag{11}$$

The first is the eigenvalue/vector problem which we used to get the first solution. We use this in the second to find $\bar{v}$. The general solution is

$$\bar{x} = c_1 \bar{u} e^{\lambda t} + c_2 \left(\bar{u} t e^{\lambda t} + \bar{v} e^{\lambda t} \right) \tag{12}$$

Complex Eigenvalues

Suppose we have eigenvalues $\lambda = \alpha \pm \beta i$ and associated eigenvectors $\bar{u} = R \pm iI$ where R and I are real vectors. Then the general solution is

$$\begin{aligned} \bar{x} &= k_1 (R + Ii) e^{\alpha t} (\cos \beta x + i \sin \beta x) + k_2 (R - Ii) e^{\alpha t} (\cos \beta x - i \sin \beta x) \\ &= (k_1 + k_2) e^{\alpha t} \Big[R \cos \beta t - I \sin \beta t \Big] + i (k_1 - k_2) e^{\alpha t} \Big[R \sin \beta t + I \cos \beta t \Big] \end{aligned} \tag{13}$$

and setting $c_1 = k_1 + k_2$ and $c_2 = i(k_1 - k_2)$ gives the solution as

$$\bar{x} = c_1 e^{\alpha t}\Big[R\cos\beta t - I\sin\beta t\Big] + c_2 e^{\alpha t}\Big[R\sin\beta t + I\cos\beta t\Big]. \tag{14}$$

Here we consider an example of each.

Example 1.

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\bar{x}$$

then the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2) = 0,$$

from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = 2$.

Case 1: $\lambda = -1$

From (3) we have

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which we obtain after expanding $2c_1 + c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Case 2: $\lambda = 2$

From (3) we have

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which we obtain after expanding $c_2 - c_1 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution to () is then given by

$$\bar{x} = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}\mathrm{e}^{-\mathrm{t}} + \mathrm{c}_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\mathrm{e}^{2\mathrm{t}}.$$

Example 2.

Consider

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}\bar{x}, \quad \bar{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

then the characteristic equation is

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0,$$

from which we obtain the eigenvalues $\lambda = 1, 1 -$ repeated.

Case 1: $\lambda = 1$

In this case we have

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $c_1 - c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so one solution is

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t.$$

For the second independent solution we seek a second solution of the form

$$\bar{x}_2 = \bar{u}te^t + \bar{v}e^t. \tag{15}$$

We already have $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and from (11)

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{16}$$

or $2v_1 - 2v_2 = 1$. Here, we'll choose

$$\bar{v} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

Therefore, the second solution is

$$\bar{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^t$$

and the general solution

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t\, e^t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^t \right],$$

Imposing the initial condition gives

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

This gives $c_1 + 1/2c_2 = 2$ and $c_1 = 1$ so $c_2 = 1$ and the general solution then becomes

$$\bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t\, e^t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^t \right],$$

Example 3.

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} \bar{x}. \tag{17}$$

The characteristic equation is

$$\begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 17 = 0.$$

Using the quadratic formula, we obtain $\lambda = 4 \pm i$ (so $\alpha = 4$ and $\beta = 1$). For the eigenvectors, we wish to solve

$$\begin{pmatrix} 5-(4+i) & 1 \\ -2 & 3-(4+i) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which means solving

$$(1-i)u + v = 0.$$

One solution is

$$\bar{u} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i.$$

So here

$$\bar{R} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \vec{I} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

With $\alpha = 4$ and $\beta = 1$ gives

$$\vec{x}_1 = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] e^{4t}, \quad \vec{x}_2 = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t \right] e^{4t}.$$

The general solution is just a linear combination of these two

$$\vec{x} = c_1 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t \right] e^{4t}.$$

Extra Example

Example 4.

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \bar{x}. \tag{18}$$

The characteristic equation is

$$\begin{vmatrix} \lambda - 6 & 1 \\ -5 & \lambda - 4 \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0.$$

Using the quadratic formula, we obtain $\lambda = 5 \pm 2i$ (so $\alpha = 5$ and $\beta = 2$). For the eigenvectors, we wish to solve

$$\begin{pmatrix} 5 + 2i - 6 & 1 \\ -5 & 5 + 2i - 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$\begin{pmatrix} -1 + 2i & 1 \\ -5 & 1 + 2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which means solving

$$-5v_1 + (1 + 2i)v_2 = 0.$$

One solution is

$$\bar{v} = \begin{pmatrix} 1 + 2i \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} i.$$

So here

$$\bar{A} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \vec{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

With $\alpha = 5$ and $\beta = 2$ gives

$$\vec{x}_1 = \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right] e^{5t}, \quad \vec{x}_2 = \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right] e^{5t}.$$

The general solution is just a linear combination of these two

$$\vec{x} = c_1 \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right] e^{5t} + c_2 \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right] e^{5t}.$$