Calculus 3 - Chain Rule

If $y = \sqrt{x^2+1}$ find $\frac{dy}{dx}$. In Calc 1 we encountered this type of problem and introduced what is know as the chain rule. If we let $u = x^2+1$, the $y = \sqrt{u}$ and found $\frac{dy}{dx}$ by calculating

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \tag{1}$$

So in this example

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}. \tag{2}$$

Chain Rule in Higher Dimensions

For functions in 2 independent variables $z = f(x,y)$ we derive the chain rule using the differential

$$dz = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy \tag{3}$$

Before doing so, we note that there are two possibilities. First suppose the

$$x = g(t), \quad y = h(t) \tag{4}$$

so

$$z = f\Big(g(t), h(t)\Big) \tag{5}$$

so $z = F(t)$ and it would make sense to have $\frac{dz}{dt}$

The other possibility is

$$x = g(t,s), \quad y = h(t,s) \tag{6}$$

so

$$z = f\Big(g(t,s), h(t,s)\Big) \tag{7}$$

so $z = F(t,s)$ and it would make sense to have $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial s}$.

Type 1 Chain Rule

Suppose that $z = f(x,y)$, $x = g(t)$, and $y = h(t)$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \tag{8}$$

Let us look at some examples.

Example 1. Consider

$$z = x^2 y \tag{9}$$

and

$$x = t + 1, \qquad s = e^t. \tag{10}$$

We first calculate the derivative directly. So

$$z = (t+1)^2 e^t \tag{11}$$

Using the product rule we obtain

$$\frac{dz}{dt} = 2(t+1)e^t + (t+1)^2 e^t. \tag{12}$$

Next we use the chain rule (8). Calculating derivatives

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2, \quad \frac{dx}{dt} = 1, \quad \frac{dy}{dt} = e^t. \tag{13}$$

From (8) we obtain

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\\ &= 2xy\cdot 1 + x^2\cdot e^t \\ &= 2(t+1)e^t+(t+1)^2e^t\end{aligned} \tag{14}$$

which we see is (12).

Example 2. Pg. 917 #8 Consider

$$w=\cos(x-y) \tag{15}$$

and

$$x=t^2, \qquad y=1. \tag{16}$$

We first calculate the derivative directly. So

$$w=\cos(t^2-1) \tag{17}$$

Using the chain rule we obtain

$$\frac{dw}{dt}=-\sin(t^2-1)\cdot 2t. \tag{18}$$

Next we use the chain rule (8). Calculating derivatives

$$\frac{\partial w}{\partial x}=-\sin(x-y), \quad \frac{\partial f}{\partial y}=\sin(x-y), \quad \frac{dx}{dt}=2t, \quad \frac{dy}{dt}=0. \tag{19}$$

From (8) we obtain

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \\
&= -\sin(x-y) \cdot 2t + \sin(x-y) \cdot 0 \\
&= -2t\sin(t^2-1)
\end{aligned} \tag{20}$$

which we see is (18).

Type 2 Chain Rule

Suppose that $z = f(x,y)$, $x = g(t,s)$, and $y = h(t,s)$.

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial s} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}
\end{aligned} \tag{21}$$

Let us look at some examples.

Example 3. Consider

$$z = x^2y^2 \tag{22}$$

and

$$x = t+s, \qquad s = t-s \tag{23}$$

We first calculate the derivatives directly. So

$$z = (t+s)^2(t-s)^2 = t^4 - 2t^2s^2 + s^4 \tag{24}$$

and the derivatives are

$$\begin{aligned}\frac{\partial z}{\partial t} &= 4t^3 - 4ts^2 \\ \frac{\partial z}{\partial s} &= -4t^2 s + 4s^3\end{aligned} \tag{25}$$

To use the chain rule (21) we need 6 derivatives. So

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2xy^2, \quad & \frac{\partial z}{\partial y} &= 2x^2 y \\ \frac{\partial x}{\partial t} &= 1, & \frac{\partial x}{\partial s} &= 1 \\ \frac{\partial y}{\partial t} &= 1, & \frac{\partial y}{\partial s} &= -1\end{aligned}$$

and from the chain rule (21) we obtain

$$\begin{aligned}\frac{\partial z}{\partial t} &= 2xy^2 \cdot (1) + 2x^2 y \cdot (1) \\ &= 2(t+s)(t-s)^2 + 2(t+s)^2(t-s) \\ \frac{\partial z}{\partial s} &= 2xy^2 \cdot (1) + 2x^2 y \cdot (-1) \\ &= 2(t+s)(t-s)^2 + 2(t+s)^2(t-s)\end{aligned} \tag{26}$$

which we see, after expanding, is (25).

Example 4. Pg. 917 #16 Consider

$$w = y^3 - 3x^2 y \tag{27}$$

and

$$x = e^s, \qquad s = e^t \quad \text{at} \quad s = -1, t = 2 \tag{28}$$

To use the chain rule (21) we need 6 derivatives. So

$$\begin{aligned} \frac{\partial w}{\partial x} &= -6xy, & \frac{\partial w}{\partial y} &= 3y^2 - 3x^2 \\ \frac{\partial x}{\partial t} &= 0, & \frac{\partial x}{\partial s} &= e^s \\ \frac{\partial y}{\partial t} &= e^t, & \frac{\partial y}{\partial s} &= 0 \end{aligned}$$

and from the chain rule (21) we obtain

$$\begin{aligned} \frac{\partial w}{\partial t} &= -6xy \cdot (0) + (3y^2 - 3x^2) \cdot (e^t) \\ &= (3e^{2t} - 3e^{2s})e^t \\ \frac{\partial w}{\partial s} &= -6xy \cdot (e^s) + (3y^2 - 3x^2) \cdot (0) \\ &= -6e^s e^t e^s \end{aligned} \tag{29}$$

Then we evaluate these at the point so

$$\frac{\partial w}{\partial t} = 3e^6 - 3, \quad \frac{\partial w}{\partial s} = -6. \tag{30}$$

More Variables

Example 5. Pg. 917 #12 Consider

$$w = xy^2 + x^2z + yz^2 \tag{31}$$

and

$$x = t^2, \quad y = 2t, \quad z = 2 \tag{32}$$

The appropriate chain rule is

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt} \tag{33}$$

These derivatives are fairly easy to calculate giving

$$\begin{aligned} \frac{dw}{dt} &= (y^2 + 2xz) \cdot 2t + (2xy + z^2) \cdot 2 + (x^2 + 2yz) \cdot 0 \\ &= 2t(4t^2 + 4t^2) + 2(4t^3 + 4) \\ &= 24t^3 + 8. \end{aligned} \tag{34}$$

Example 6. Pg. 917 #22 Consider

$$w = x\cos(yz) \tag{35}$$

and

$$x = s^2, \qquad y = t^2, \quad z = s - 2t \tag{36}$$

The appropriate chain rule is

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s} \\ \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t} \end{aligned} \tag{37}$$

Here we calculate only the first one

$$\begin{aligned} \frac{\partial w}{\partial s} &= \cos(yz) \cdot 2s - xz\sin(yz) \cdot (0) - xy\sin(yz) \cdot (1) \\ &= 2s\cos\left(t^2(s-2t)\right) - t^2 s^2 \sin\left(t^2(s-2t)\right) \end{aligned} \tag{38}$$