Math 3331 - ODE's

Ordinary Differential Equations - (ODEs)

So what are they anyway? Simply put - an equation which involves ordinary derivatives

Ex If $y = f(x)$

then $y' = y$ is an example

so is $y' + y = 0$, $y'' + y = 0$ $xy' + y = 1$

these are all examples.

The goal of this course is, given an ODE, find a solution - a function such that when it is substituted into the ODE it is automatically satisfied.

Ex $y' - y = 0$

Sol^n $y = e^x$ since $y' = e^x$ &

$LS = y' - y = e^x - e^x = 0 = R.S.$

In fact $y = ce^x$ will work (c arbitrary constant)

Some terminology

order - the highest derivative in the ODE determines the order

$y' + y = 0$ 1st order

$y'' + y = 0$ 2nd order

$y^{(4)} + xy = 0$ 4th order

when possible, we usually isolate the highest derivative so

$xy' + y = 0 \Rightarrow y' = -y/x$

$xy'' + y' + y = 0 \Rightarrow y'' = -\dfrac{y' + y}{x}$

IF your 1st order ODE can be written as

$a(x)y' + b(x)y = g(x)$

$a(x)y'' + b(x)y' + c(x)y = g(x)$

$\vdots$

these are referred to as linear as the ODE is linear in $y, y', \ldots$

$y'' + \sin(y) = 0$

nonlinear

Unfortunately, most real world ODEs are nonlinear if $g(x) \equiv 0$ we say

$a(x)y' + b(x)y = 0$ — homogeneous

$a(x)y'' + b(x)y' + c(x)y = 0$

$\vdots$

if $g(x) \neq 0$ then the ODE is non homogeneous

so $xy' + y = 0$ — homo

$xy' + y = x^2$ non homo.

<u>Solution</u>

Given an nth order ODE

$$y^{(n)} = F(x, y, y', y'' \cdots y^{(n-1)})$$

$y^{(n)}$ nth order derivative

Then the sol^n is some function $y = \phi(x)$ that when substituted into the ODE shows its automatically satisfied. Often there will be some interval associated with the sol^n

Ex show $y = \frac{1}{x}$ is a sol^n of $y' = -y^2$

BTW — is nonlinear & 1st order

to verify we consider the left hand side 1-4
and right hand side

so $y = \frac{1}{x} = x^{-1}$ $y' = -1\, x^{-2} = -\frac{1}{x^2}$

L.S. $= y' = -\frac{1}{x^2}$ R.S. $-y^2 = -\left(\frac{1}{x}\right)^2 = -\frac{1}{x^2}$

and clearly LS = RS.

Here we consider the sol^n $(-\infty, 0)$ or $(0, \infty)$

ex verify $y = c x e^x$ is a sol^n to

$y' = \frac{y}{x} + y$

LS. $y' = ce^x + cxe^x$

RS $\frac{y}{x} + y = \frac{cxe^x}{x} + cxe^x = ce^x + cxe^x$

so LS = RS.

OR $y' - \frac{y}{x} - y = 0$

LS $= ce^x + cxe^x - \frac{c\cancel{x}e^x}{\cancel{x}} - cxe^x$

$= 0 =$ RS.

One will notice that in the previous ex.

ODE $y' = \frac{y}{x} + y$

the sol^n was $y = cxe^x$ - there is an arbitrary constant c in the sol^n

This is often the case - constants that appear in the sol^n. So

$y = cxe^x$ is the sol^n & $y = -3xe^x$ is just a particular case and refered to as a particular sol^n.

In general nth order ODE's will have n arbitrary consts

Ex Verify $y = c_1 \sin x + c_2 \cos x$ is a sol^n

$y'' + y = 0$

$y = c_1 \sin x + c_2 \cos x$, $y' = c_1 \cos x - c_2 \sin x$

$y'' = -c_1 \sin x - c_2 \cos x$

LS =

$y'' + y = -c_1 \sin x - c_2 \cos x + c_1 \sin x + c_2 \cos x$

$= 0 = RS$ ✓

So how do we find c_1 & c_2.
We need more information. So we may be given

ii) $y(0) = 0, \quad y'(0) = 1$ some values of y at $x=0$

(Called an initial value problem IVP)

iii) $y(0) = 0, \quad y(\pi/2) = 1$
Some values on an interval of "boundary"
and this is called ~~an~~ a boundary value prob (BVP)

<u>Previous Ex</u>
find c_1 & c_2 if $y(0) = 0$, $y'(0) = 1$

$y = c_1 \sin x + c_2 \cos x \qquad y(0) = c_1 \sin(0) + c_2 \cos(0) = 0$
$\qquad\qquad\qquad\qquad\qquad = 0 + c_2 = 0 \Rightarrow c_2 = 0$

$y' = c_1 \cos x \qquad y'(0) = c_1 \cos(0) = 1 \Rightarrow c_1 = 1$
(note $c_2 = 0$)
sol^n subject to the IC's is

$y = \sin x.$

<u>Now</u> How do we find the sol^n's?