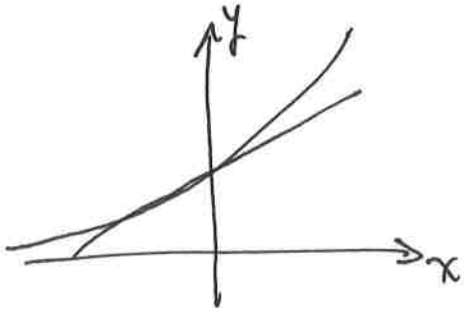
Math 1492 – Calc 2

9.1 Taylor Polynomials

As we have been introduced to power series we now want a way to generate these ↑.

Consider $y = e^x$. Let us approximate this function using a linear function at $x = 0$

(looks like tangent)



$P_1(x) = c_0 + c_1 x \qquad f(x) = e^x$

(i) we want $f(0) = P_1(0) \Rightarrow \quad c_0 + c_1(0) = e^0 \Rightarrow c_0 = 1$

(ii) slopes to agree

$f'(0) = P_1'(0) \Rightarrow \quad c_1 = e^0 \Rightarrow c_1 = 1$

so $\quad P_1(x) = 1 + x$

Tangent $\quad y = f(a) + f'(a)(x-a) \qquad f' = e^x \quad f'(0) = 1$

$= 1 + 1(x-0) = 1 + x \quad$ same

For a better approximation we go to 2nd degree polynomials (parabolas)

$$P_2(x) = c_0 + c_1 x + c_2 x^2$$

we want (i) $f(0) = P_2(0)$ at pt

(ii) $f'(0) = P_2'(0)$ slopes

(iii) $f''(0) = P_2''(0)$ concavity

$$P_2(0) = c_0 + c_1 \cdot 0 + c_2 \cdot 0 = f(0) = e^0 = 1$$

$$\Rightarrow c_0 = 1$$

$$P_2'(x) = c_1 + 2c_2 x \qquad f' = e^x$$

$$P_2'(0) = c_1 + 2c_2(0) = f'(0) = e^0$$

$$\Rightarrow c_1 = 1$$

$$P_2''(x) = 2c_2 \qquad f'' = e^x$$

$$P_2''(0) = f''(0) \Rightarrow 2c_2 = 1 \Rightarrow c_2 = 1/2$$

$$P_2(x) = 1 + x + \frac{1}{2}x^2$$

3rd order

$$P_3 = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \qquad f(x) = e^x$$

(i) $P_3(0) = f(0) \Rightarrow c_0 = 1$

(ii) $P_3'(0) = f'(0) \Rightarrow c_1 + 2c_2(0) + 3c_3(0)^2 = e^0$

$\Rightarrow c_1 = 1$ Same

(iii) $P_3''(0) = f''(0) \Rightarrow 2c_2 + 3\cdot 2 c_3(0) = e^0$

$\Rightarrow c_2 = 1/2$

iv) $P_3^{(4)}(0) = f^{(4)}(0)$

$$P_3^{(4)} = 3\cdot 2\cdot 1\, c_3 \qquad f^{(4)}(x) = e^x$$

$$P_3^{(4)}(0) = f^{(4)}(0) \Rightarrow 3\cdot 2 c_3 = 1 \Rightarrow c_3 = \frac{1}{3!}$$

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

once we see the pattern

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}$$

nth degree Taylor Poly for $y = e^x$

if we let $n \to \infty$ we get a Taylor Series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{and it converges to} \quad e^x$$

famous series

Taylor Polynomials in general

Given $y = f(x)$

The Taylor Polynomial at $x = a$
n^{th} order

$$P_1 = f(a) + f'(a)(x-a) \qquad 1^{st}$$

$$P_2 = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$$

$$\vdots$$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Note: The n^{th} degree Taylor Poly. will need n derivative of $f(x)$!

Ex 1 Find $P_5(x)$ for $f(x) = \sin x$ at $x = 0$

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1$$

$$P_5(x) = 0 + \frac{1(x-0)}{1!} + 0\frac{(x-0)^2}{2!} - \frac{(x-0)^3}{3!} + 0\frac{(x-0)^4}{4!} + 1\frac{(x-0)^5}{5!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

we see pattern powers skip by 2's
the series alternates and terms / $n!$

$$P_{11} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{Taylor series}$$

Ex2 Find $P_4(x)$ for $f(x) = \frac{1}{x}$ at $x = 1$

$f(x) = x^{-1}$ $\qquad f(1) = 1$

$f'(x) = -1\,x^{-2} = -\frac{1}{x^2}$ $\qquad f'(1) = -1$

$f''(x) = 2x^{-3} = \frac{2}{x^{-3}}$ $\qquad f''(1) = 2$

$f'''(x) = -2 \cdot 3 x^{-4} = -\frac{2 \cdot 3}{x^4}$ $\qquad f'''(1) = -2 \cdot 3$

$f^{(4)}(x) = 2 \cdot 3 \cdot 4\, x^{-5} = \frac{2 \cdot 3 \cdot 4}{x^5}$ $\qquad f^{(4)}(1) = 2 \cdot 3 \cdot 4$

$$P_4(x) = 1 - 1(x-1) + \frac{2(x-1)^2}{2!} - \frac{2 \cdot 3 (x-1)^3}{3!} + \frac{2 \cdot 3 \cdot 4 (x-1)^4}{4!}$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$$

Ex3 Find $P_3(x)$ if $f(x)=\sqrt{x}$ and $a=4$

$f(x) = x^{1/2}$ $\qquad f(4) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2} x^{-1/2}$ $\qquad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-3/2}$ $\qquad f''(4) = -\frac{1}{4} \frac{1}{(4)^{3/2}} = -\frac{1}{4} \cdot \frac{1}{8}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-5/2}$ $\qquad f'''(4) = \frac{3}{8} \cdot \frac{1}{32}$

$$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32} \frac{(x-4)^2}{2!} + \frac{3}{8\cdot 32} \frac{(x-4)^3}{3!}$$

$$= 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{1}{512} \cdot (x-4)^3$$