Limits

1) $\lim_{x \to 4} \dfrac{4(x-4)}{\sqrt{x}-2} \Rightarrow$ when the is a $\sqrt{\ }$ it's best to try to get rid of it

$$\frac{4(x-4)}{\sqrt{x}-2} * \underbrace{\frac{\sqrt{x}+2}{\sqrt{x}+2}}_{\text{switch sign}} = \frac{4\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}} = 4(\sqrt{x}+2)$$

$$\lim_{x \to 4} 4(\sqrt{x}+2) = 4(\sqrt{4}+2)$$

$$= 16 \ !$$

Other important limits

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0}$$ use l'Hopital's Rule

$$\lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = \frac{1}{1} = 1$$

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

in general $\lim_{x \to 0} \frac{\sin ax}{x} \cong \frac{ax}{x} = a$

Ex: $\lim_{x \to 0} \frac{\tan 2x}{x} \Rightarrow \lim_{x \to 0} \frac{\frac{\sin 2x}{\cos 2x}}{x}$

$$\Rightarrow \frac{\sin 2x}{x} \cdot \frac{1}{\cos 2x} = 2 \cdot \frac{1}{\cos 2x}$$

$$\Rightarrow \frac{2}{\cos 0} = 2.$$

$$\boxed{\lim_{x \to 0} \frac{\tan 2x}{x} = 2}$$

#61) $f(x) = x^3 + \frac{243}{x}$ $\quad f'(x) = 3x^2 - \frac{243}{x^2}$

extremums:

$f'(x) = 0 = 3x^2 - \frac{243}{x^2} \Rightarrow x^4 = \frac{243}{3} = 81$ $\quad \boxed{x = \pm 3}$

To find what kind of point use 2nd derivative

$f''(x) = 6x + \frac{2 \cdot 243}{x^3}$

for $x = +3$

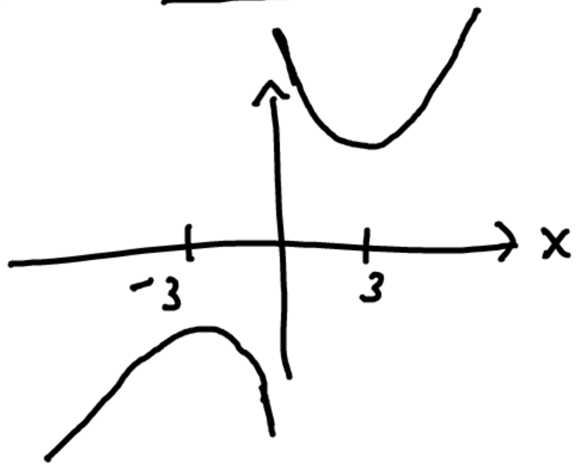
$f''(3) > 0$ therefore $x = +3$ is a <u>minima</u>

$x = -3$ is a <u>maximum</u>

$f''(-3) < 0$

for $x \Rightarrow 0^+$ $\quad f(x) = +\infty$; $f'(x) = -\infty$

$x \Rightarrow 0^-$ $\quad f(x) = -\infty$



63) $f(x) = \dfrac{x-1}{1+3x^2}$ this is a little harder but always the same

$$f'(x) = \frac{(1+3x^2) - (x-1)(6x)}{(1+3x^2)^2} = \frac{-3x^2+6x+1}{(1+3x^2)}$$

$$f'(x) = 0 = -3x^2+6x+1 \qquad x = \frac{-6 \pm \sqrt{6^2+4\cdot 3\cdot 1}}{2\cdot(-3)}$$

$$\boxed{x = 1 \pm \frac{2}{\sqrt{3}}}$$

$f''(x) \Rightarrow$ quite long $\rightarrow$ use DESMOS!!!

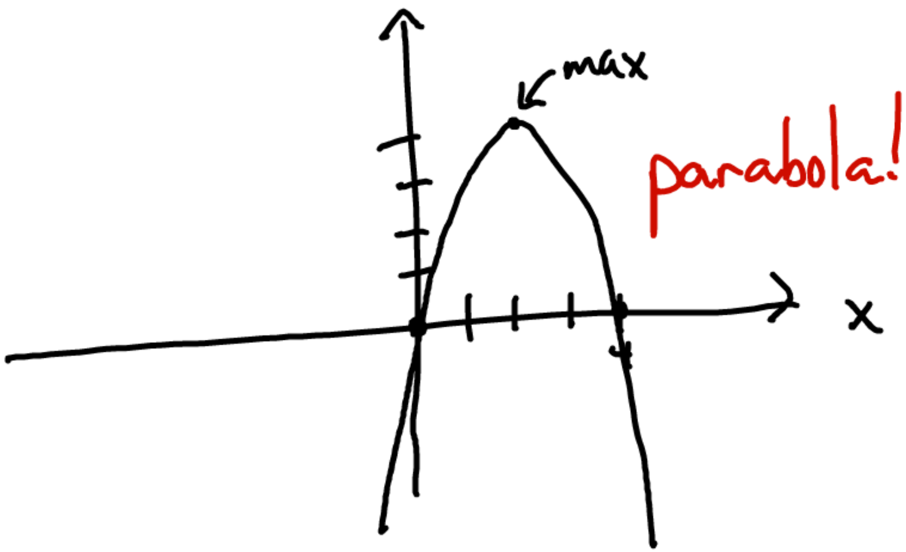
65) $f(x) = 4x - x^2$; $f'(x) = 4 - 2x$: $f''(x) = -2$

1) where does $f(x) = \phi$ $\quad 0 = x(4 - x)$ $\quad x = 0$, $x = +4$ $\quad f(x) = 0$

2) find extremums

$f'(x) = \phi = 4 - 2x$ $\quad \boxed{x = +2}$ $\quad f(2) = 8 - 4 = 4$

3) $f''(2) = -2 \Rightarrow$ since $f'' < 0$ we know that $x = +2$ is a maximum. concave down.



parabola!!

67) $f(x) = x\sqrt{16-x^2}$ $\qquad f(x) = \phi \Rightarrow x = \phi$

$x = \pm 4$

Also note that when $|x| > 4$ function does not exist

$$f'(x) = \sqrt{16-x^2} + x \cdot \frac{1}{2} \frac{1}{\sqrt{16-x^2}} (-2x)$$

$$f'(x) = \sqrt{16-x^2} - \frac{x^2}{\sqrt{16-x^2}} \Rightarrow f'(x) = \frac{1-x^2}{\sqrt{16-x^2}}$$

Find extremums $f'(x) = \phi$

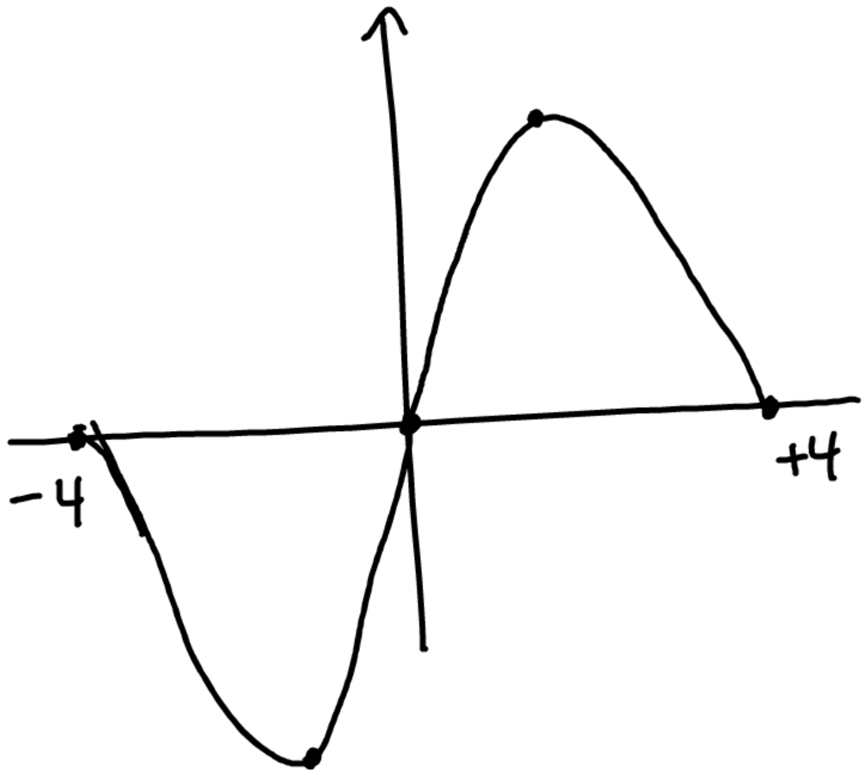
$f'(x) = \phi$ when $\boxed{x = \pm 1}$ $\qquad$ we only need to know if $f''(x)$ is +ve or −ve.

$$f''(x) = \frac{-2x\sqrt{16-x^2} - (1-x^2)\frac{1}{2}\frac{-2x}{\sqrt{16-x^2}}}{(16-x^2)}$$

this term = 0 at $x = \pm 1$

at $x = +1$ $f''(x) < 0 \Rightarrow$ max; at $x = -1$; $f''(x) > 0 \Rightarrow$ minimum.

sketch of $f(x)$



73) $f(x) = \dfrac{5-3x}{x-2}$;

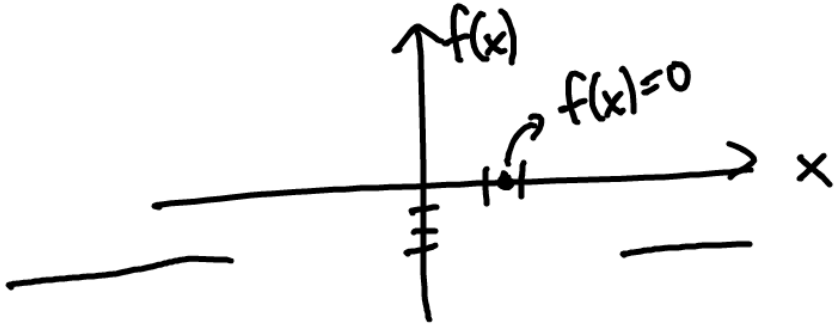
1) when is $f(x)=0$ $\quad 5-3x=0 \quad \boxed{x=\frac{5}{3}}$

2) when $x \to \infty$ what does it look like

$$\lim_{x\to\infty} \frac{5-3x}{x-2} = \frac{\cancel{x}\left(\frac{5}{x}-3\right)}{\cancel{x}\left(1-\frac{2}{x}\right)} = -3$$

$$\lim_{x\to-\infty} \frac{5-3x}{x-2} \longrightarrow -3$$

So when we are far $x = \pm\infty$; $f(x) = -3$



Now we fill in the rest.

73) continue $f(x) = \dfrac{5-3x}{x-2}$

Note that when $x=2$ we have a vertical asymptote

$$f(2) \to \infty$$

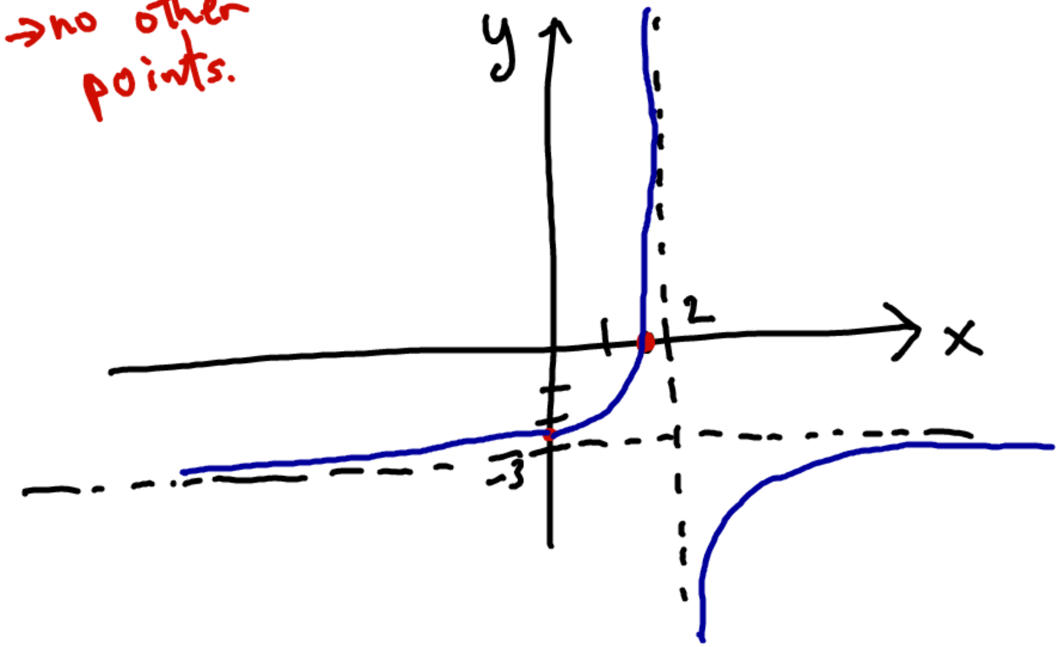
$$f'(x) = \frac{-3(x-2) - (5-3x)}{(x-2)^2} = \frac{6-5}{(x-2)^2} = \frac{1}{(x-2)^2}$$

when $x \to \pm\infty$ $f'(x) \to \phi$ → no other points.

$f'(x)$ is always positive.

Also

$f(0) = -5/2$



77) $f(x) = x^3 + x + \frac{4}{x}$

when $x = 0^+$ $f(0^+) \to \infty$ $\Rightarrow$ vertical asymptote

$x = 0^-$ $f(0^-) \to -\infty$

$f'(x) = 3x^2 + 1 - \frac{4}{x^2}$ $\quad f'(x) = 0 = 3x^2 + 1 - \frac{4}{x^2}$

$0 = 3x^4 + x^2 - 4$ $\quad \frac{-1 \pm \sqrt{1 + 4 \cdot 3 \cdot 4}}{6}$

$\frac{-1 \pm \sqrt{49}}{6} \Rightarrow \frac{-1 \pm 7}{6} \Rightarrow -\frac{8}{6}, \frac{6}{6} = -\frac{4}{3}, 1$

$x^2 = -\frac{4}{3}, 1$

Real solutions $x^2 = 1 \Rightarrow x = \pm 1$

At $x \to 0^+$ $f'(x) \to -\infty$
$x \to 0^-$ $f'(x) \to -\infty$
$\Big\}$ negative

$$f'(x) = 3x^2 + 1 - \frac{4}{x^2}$$

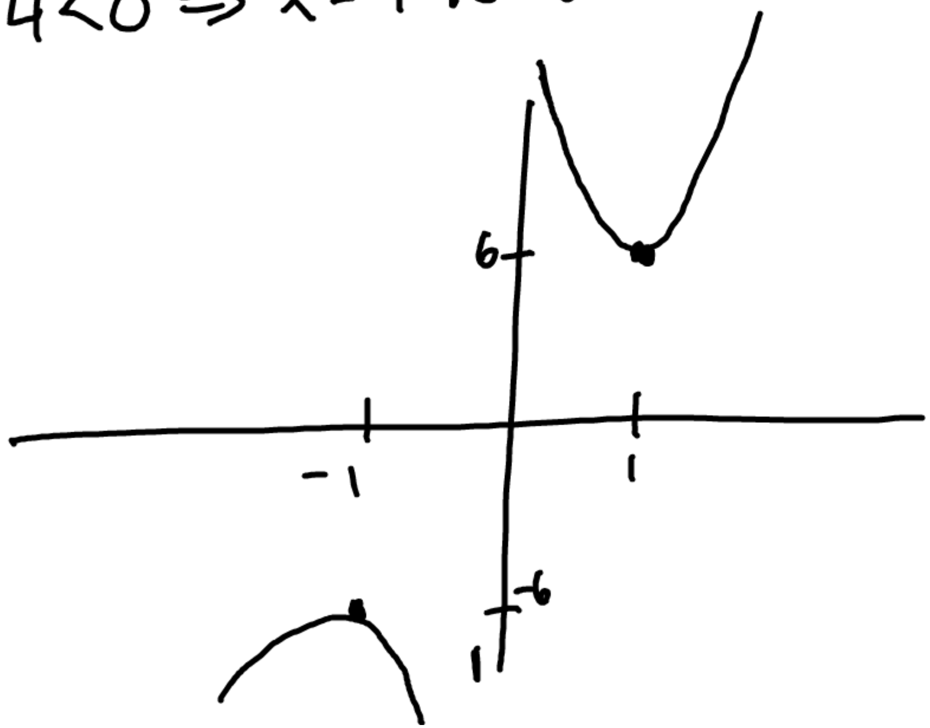
$f(1) = 6$; $f(-1) = -6$

$$f''(x) = 6x + \frac{4}{x^3}$$

$f''(1) = 6 + \frac{4}{1^3} > 0 \Rightarrow x = 1$ is a minimum

$f''(-1) = -6 - 4 < 0 \Rightarrow x = -1$ is a maximum

Try in Desmos!!!



Define which function is $f(x)$, $f'(x)$, $f''(x)$?